Sensitivity analysis of a Tresca-type problem leads to Signorini's conditions

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January 25, 2022

Abstract

The present paper investigates the sensitivity analysis, with respect to right-hand source term perturbations, of a scalar Tresca-type problem. This simplified, but nontrivial, model is inspired from the (vectorial) Tresca friction problem found in contact mechanics. The weak formulation of the considered problem leads to a variational inequality of the second kind depending on the perturbation parameter. The unique solution to this problem is then characterized by using the proximal operator of the corresponding nondifferentiable convex integral friction functional. We compute the convex subdifferential of the friction functional on the Sobolev space $\mathrm{H}^1(\Omega)$ and show that all its subgradients satisfy a PDE with a boundary condition involving the convex subdifferential of the integrand. With the aid of the *twice epi-differentiability*, concept introduced and thoroughly studied by R.T. Rockafellar, we show the differentiability of the solution to the parameterized Tresca-type problem and that its derivative satisfies a Signorini-type problem. Some numerical simulations are provided in order to illustrate our main theoretical result. To the best of our knowledge, this is the first time that the concept of twice epi-differentiability is applied in the context of mechanical contact problems, which makes this contribution new and original in the literature.

Keywords: Tresca-type problem; Signorini-type problem; variational inequality; convex subdifferential; proximal operator; sensitivity analysis; twice epi-differentiability.

AMS Classification: 49Q12, 46N10, 74M15.

1 Introduction

Mechanical context. Contact and friction phenomena with deformable bodies are increasingly taken into account in industrial models and engineering applications. We can cite for example: wheel-ground contact analysis in aeronautics, assemblies of mechanical processes, modeling of medical prostheses, etc. In general the mechanical setting consists in a deformable body which is in contact with a rigid foundation. The elastic body is deformed under some volume forces and surface tractions without penetrating the rigid foundation. Usually the mathematical models of

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these mechanical contact problems lead to nonlinear boundary value problems, including unilateral (possibly nonsmooth) constraints, where the unknowns are the displacement and the stress field. The corresponding weak mathematical formulations are expressed as variational inequalities of the first or second kind. These variational formulations are usually used to prove existence, uniqueness, regularity of solutions as well as for numerical purposes.

The so-called Signorini problem is a mechanical contact problem without friction. It consists in finding the equilibrium configuration of an elastic body in a frictionless contact with a rigid surface. This problem was first formulated by A. Signorini [33] in 1933, and later in 1959 in the paper [34]. In 1963, G. Fichera proved in [12] the existence and uniqueness of the solution to the Signorini problem by minimizing the corresponding quadratic potential energy functional. Signorini's laws are expressed as complementarity relations and translate the non-penetrability of the contact zone on the obstacle, the non-appearance of traction forces on the contact zone and the complementarity of normal forces and displacements. The weak formulation of the Signorini problem can be recast into a variational inequality of the first kind and the literature is abundant on both theoretical and numerical aspects on this subject. Comprehensive references in this field include [3, 4, 21, 31]. When dealing with frictional contact problems of deformable bodies, a Coulomb friction model was studied by G. Duvaut and J.-L. Lions [10]. The main difficulty of this model comes from the fact that the friction functional depends on the normal stress of the unknown displacement, which leads to a nonvariational problem. The Tresca model can be seen as a simplified Coulomb friction law with a given friction bound or threshold (see, e.g., [10]). It can be considered as a first step towards the treatment of the more complicated mathematical formulation of the Coulomb friction law. The weak formulation of the Tresca friction problem is a variational inequality of the second kind involving a nondifferentiable convex integral friction functional. For more details about the formulations of the Tresca and Coulomb models, we refer the reader to [10, 20, 32].

Motivations. In general optimization theory, the sensitivity analysis of the state with respect to given parameters plays a fundamental role in order to formulate necessary optimality conditions or for numerical purposes (for the implementation of gradient descent methods for example). When we started this collaboration, our primary motivation was shape optimization problems, that is determining the optimal design of a given object for industrial or engineering applications, involving mechanical contact and friction phenomena. We gradually unrolled all the underlying issues that allow us to prepare the ground for the treatment of such problems. In particular dealing with the sensitivity analysis of the state of such problems is a difficult task due to the unilateral (possibly nonsmooth) character of the models mentioned in the previous paragraph.

The aim of the present work is to provide an original methodology based on mathematical tools from convex and variational analyses in order to deal with the sensitivity analysis of a Tresca-type problem with respect to right-hand source term perturbations. Precisely, in this paper, we focus on a scalar version of the (vectorial) Tresca friction problem. This scalar version can be found in previous works such as [14, Section 5 Chapter 2], [15, Section 1.3 Chapter 1] or more recently in [16, Chapter 5] and [35, Chapter 9]. Hence the present work constitutes a nontrivial first step towards the treatment of the (vectorial) Tresca friction problem found e.g. in [10]. In the whole paper, by analogy with this model in contact mechanics, we decided to refer to the considered problem as "Tresca-type problem". Note that we will be using later a similar terminology by using "Signorini-type problem".

In this paper we deal with the scalar Tresca-type problem given by

$$\left\{\begin{array}{rl} -\Delta u+u=f & \text{in } \Omega, \\ |\partial_{\mathrm{n}} u| \leq 1 \text{ and } u\partial_{\mathrm{n}} u=-|u| & \text{on } \Gamma, \end{array}\right. \tag{TP}$$

where $\Omega \subset \mathbb{R}^d$ is a nonempty bounded connected open subset of $\mathbb{R}^d$, $d \in \mathbb{N}^*$ (where $\mathbb{N}^*$ stands for the set of positive integers), with a Lipschitz continuous boundary $\Gamma := \partial\Omega$ and $f \in \mathrm{L}^2(\Omega)$. Considering that the right-hand source term f is perturbed and replaced by $f_t \in \mathrm{L}^2(\Omega)$, where $t \geq 0$ is a parameter, our aim is to study the differentiability at $t = 0$ of the unique solution u_t to the parameterized problem (obtained by replacing f with f_t, see Problem (TP$_t$) below) and to express its derivative, denoted by u_0', as the unique solution to a new boundary value problem.

Main result. The main result of this paper claims that, for a given right-hand source term $f_t \in \mathrm{L}^2(\Omega)$ depending on a parameter $t \geq 0$, the map $t \geq 0 \longmapsto u_t \in \mathrm{H}^1(\Omega)$, where u_t stands for the unique solution to the parameterized Tresca-type problem

$$\left\{\begin{array}{rl} -\Delta u_t+u_t=f_t & \text{in } \Omega, \\ |\partial_{\mathrm{n}} u_t| \leq 1 \text{ and } u_t\partial_{\mathrm{n}} u_t=-|u_t| & \text{on } \Gamma, \end{array}\right. \tag{TP$_t$}$$

is differentiable at $t = 0$, and its derivative $u_0' \in \mathrm{H}^1(\Omega)$ is the unique solution to the Signorini-type problem

$$\left\{\begin{array}{rl} -\Delta u_0'+u_0'=f_0' & \text{in } \Omega, \\ \partial_{\mathrm{n}} u_0'=0 & \text{on } \Gamma_{\mathrm{N}}^{u_0}, \\ u_0'=0 & \text{on } \Gamma_{\mathrm{D}}^{u_0}, \\ u_0' \leq 0,\ \partial_{\mathrm{n}} u_0' \leq 0 \text{ and } u_0'\partial_{\mathrm{n}} u_0'=0 & \text{on } \Gamma_{\mathrm{S}-}^{u_0}, \\ u_0' \geq 0,\ \partial_{\mathrm{n}} u_0' \geq 0 \text{ and } u_0'\partial_{\mathrm{n}} u_0'=0 & \text{on } \Gamma_{\mathrm{S}+}^{u_0}, \end{array}\right. \tag{SP$_0'$}$$

where the decomposition of the boundary $\Gamma = \Gamma_{\mathrm{N}}^{u_0} \cup \Gamma_{\mathrm{S}-}^{u_0} \cup \Gamma_{\mathrm{D}}^{u_0} \cup \Gamma_{\mathrm{S}+}^{u_0}$ depends on u_0 (see Theorem 3.16 for details). This result, proved under some appropriate assumptions, establishes a direct link between scalar Tresca-type and Signorini-type problems. Precisely, in our context, it emphasizes the fact, roughly speaking, that solutions to Signorini-type problems can be considered as first-order approximations of perturbed solutions to Tresca-type problems in the following sense: for small values $t > 0$, the function u_t can be approximated in H^1-norm by $u_0 + tu_0'$. Such approximations are numerically computed on some explicit examples at the end of the present manuscript for an illustrative purpose of the main theoretical result.

Methodology. Our methodology is based on mathematical tools from convex analysis. First, the weak formulation of Problem (TP$_t$) is given by the following variational inequality of the second kind

$$\int_\Omega \nabla u_t \cdot \nabla(\varphi - u_t) + \int_\Omega u_t(\varphi - u_t) + \Phi(\varphi) - \Phi(u_t) \geq \int_\Omega f_t(\varphi - u_t),$$

for all $\varphi \in \mathrm{H}^1(\Omega)$, where Φ stands for the Tresca-type functional (which is a proper lower semi-continuous and convex function) given by

$$\begin{array}{rccl} \Phi : & \mathrm{H}^1(\Omega) & \longrightarrow & \mathbb{R} \\ & w & \longmapsto & \Phi(w) := \displaystyle\int_\Gamma |w|\,. \end{array}$$

It follows that the unique solution to Problem (TP_t) can be expressed in terms of Moreau's *proximal operator* prox_Φ of Φ (see [24, 25] and Section 2.2 for some reminders) as

$$u_t = \mathrm{prox}_\Phi(F_t),$$

for all $t \geq 0$, where $F_t \in \mathrm{H}^1(\Omega)$ stands for the unique solution to the classical Neumann problem

$$\int_\Omega \nabla F_t \cdot \nabla \varphi + \int_\Omega F_t \varphi = \int_\Omega f_t \varphi,$$

for all $\varphi \in \mathrm{H}^1(\Omega)$. Hence the differentiability of the map $t \geq 0 \longmapsto u_t \in \mathrm{H}^1(\Omega)$ at $t = 0$ is related to the differentiability (in a generalized sense) of the proximal operator prox_Φ. To this aim we use an approach based on the notion of *twice epi-differentiability* introduced by R.T. Rockafellar in [28] and characterize the derivative $u_0' \in \mathrm{H}^1(\Omega)$ in terms of the proximal operator of the second-order epi-derivative $\mathrm{d}_e^2\Phi$ of Φ, precisely as

$$u_0' = \mathrm{prox}_{\mathrm{d}_e^2\Phi(u_0|F_0-u_0)}(F_0').$$

We finally prove that the above equality also characterizes the unique solution to the Signorini-type problem (SP_0').

Additional comments. Our main result is based on the assumption of twice epi-differentiability of the Tresca-type functional Φ. Some sufficient conditions on u_0 and Γ are provided in Remark 3.18 and Appendix B in order to satisfy this hypothesis. The general setting of the twice epi-differentiability of Φ over the Sobolev space $\mathrm{H}^1(\Omega)$ remains an open challenge.

In order to illustrate our main theoretical result we provide some numerical simulations. The idea is to solve numerically the above parameterized Tresca-type problem (TP_t) for small values $t \geq 0$ and the Signorini-type problem (SP_0'). Then we compare in H^1-norm the function u_t with its first-order approximation $u_0 + tu_0'$. In order to approximate the Signorini-type problem, we use the iterative switching algorithm introduced by J.M. Aitchison and M.W. Poole in [2], due to its simplicity and to its advantage to be used in combination with other numerical methods with minimal coding development. Then we propose a revisited iterative switching algorithm for the numerical resolution of the parameterized Tresca-type problem.

For simplicity, in this paper, the boundary condition in the Tresca-type problem (TP_t) has a constant friction threshold equal to 1. In our opinion no difficulty would arise by extending our approach to a general friction threshold $g \in \mathrm{L}^2(\Gamma)$ with $g \geq 0$. Nevertheless an interesting perspective for further research works is to consider a perturbed friction threshold $g_t \in \mathrm{L}^2(\Gamma)$ depending on the parameter $t \geq 0$. In that situation the Tresca-type functional Φ_t would also be perturbed and the treatment would require some special adjustments in the definition of the twice epi-differentiability (see [1] for more details).

Organization of the paper. The paper is organized as follows. In Section 2, some useful definitions and results from convex analysis are recalled. We focus especially on the notion of twice epi-differentiability which plays a crucial role in the present paper. In Section 3, we provide all necessary ingredients in order to prove our main result, which claims that the derivative of the parameterized Tresca-type problem satisfies Signorini's conditions. Numerical simulations are provided in Section 4 for illustration of the main theoretical result. Concluding remarks are presented in Section 5. Finally some technical details are provided in the Appendices. In Appendix A, a classical result on the extension of linear operators is recalled. In Appendix B, we provide some sufficient conditions that guarantee the twice epi-differentiability of the Tresca-type functional. Appendix C is devoted to some details concerning the numerical algorithms used in Section 4.

2 Basics of convex analysis and twice epi-differentiability

This section is dedicated to recall different basic convex analysis and twice epi-differentiability results useful in Section 3 in order to establish the main result (Theorem 3.16). In this section we denote by $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$ and by H a given real Hilbert space endowed by the scalar product $\langle \cdot, \cdot \rangle_{\mathrm{H}}$ and the associated norm $\| \cdot \|_{\mathrm{H}}$. The notation $\mathrm{Id} : \mathrm{H} \to \mathrm{H}$ stands for the usual identity map. Finally, in the whole section, all limits with respect to $\tau > 0$ will be considered for $\tau \to 0^+$. For the ease of notations, when no confusion occurs, the notation $\tau \to 0^+$ will be omitted.

2.1 Mosco epi-convergence

Let $(S_\tau)_{\tau>0}$ be a parameterized family of subsets of H. The *outer*, *weak-outer*, *inner* and *weak-inner limits* of $(S_\tau)_{\tau>0}$ when $\tau \to 0^+$ are respectively defined by

$$\begin{aligned}
\limsup S_\tau &:= \{x \in \mathrm{H} \mid \exists (t_n)_n \to 0^+,\ \exists (x_n)_n \to x,\ \forall n \in \mathbb{N},\ x_n \in S_{t_n}\},\\
\text{w-}\limsup S_\tau &:= \{x \in \mathrm{H} \mid \exists (t_n)_n \to 0^+,\ \exists (x_n)_n \rightharpoonup x,\ \forall n \in \mathbb{N},\ x_n \in S_{t_n}\},\\
\liminf S_\tau &:= \{x \in \mathrm{H} \mid \forall (t_n)_n \to 0^+,\ \exists (x_n)_n \to x,\ \exists N \in \mathbb{N},\ \forall n \geq N,\ x_n \in S_{t_n}\},\\
\text{w-}\liminf S_\tau &:= \{x \in \mathrm{H} \mid \forall (t_n)_n \to 0^+,\ \exists (x_n)_n \rightharpoonup x,\ \exists N \in \mathbb{N},\ \forall n \geq N,\ x_n \in S_{t_n}\},
\end{aligned}$$

where $\to$ (respectively $\rightharpoonup$) denotes the strong (respectively weak) convergence in H. Note that the four inclusions

$$\liminf S_\tau \subset \limsup S_\tau \subset \text{w-}\limsup S_\tau \quad \text{and} \quad \liminf S_\tau \subset \text{w-}\liminf S_\tau \subset \text{w-}\limsup S_\tau,$$

always hold true.

Definition 2.1 (Mosco-convergence)**.** A parameterized family $(S_\tau)_{\tau>0}$ of subsets of H is said to be *Mosco-convergent* if

$$\text{w-}\limsup S_\tau \subset \liminf S_\tau.$$

In that case we write $\text{M-}\lim S_\tau := \liminf S_\tau = \limsup S_\tau = \text{w-}\liminf S_\tau = \text{w-}\limsup S_\tau$.

The *domain* and the *epigraph* of an extended real-valued function $\Phi : \mathrm{H} \to \overline{\mathbb{R}}$ defined on H are respectively given by

$$\mathrm{dom}(\Phi) := \{x \in \mathrm{H} \mid \Phi(x) < +\infty\} \quad \text{and} \quad \mathrm{Epi}(\Phi) := \{(x, \lambda) \in \mathrm{H} \times \mathbb{R} \mid \Phi(x) \leq \lambda\}.$$

Recall that the set of all epigraphs on H is stable under outer and inner limits (see, e.g., [30, p.240]).

Definition 2.2 (Mosco epi-convergence)**.** A parameterized family $(\Phi_\tau)_{\tau>0}$ of extended real-valued functions defined on H is said to be *Mosco epi-convergent* if $(\mathrm{Epi}(\Phi_\tau))_{\tau>0}$ is Mosco-convergent. In that case, we denote by $\text{ME-lim}\,\Phi_\tau : \mathrm{H} \to \overline{\mathbb{R}}$ the extended real-valued function defined on H characterized by its epigraph as follows:

$$\mathrm{Epi}(\text{ME-lim}\,\Phi_\tau) := \text{M-lim}\,\mathrm{Epi}(\Phi_\tau).$$

Recall the following characterization of Mosco epi-convergence. We refer for instance to [5, Proposition 3.19 p.297] or [30, Proposition 7.2 p.241] for details.

Proposition 2.3. *Let Φ be an extended real-valued function defined on H and let $(\Phi_\tau)_{\tau>0}$ be a parameterized family of extended real-valued functions defined on H. Then $(\Phi_\tau)_{\tau>0}$ Mosco epi-converges with $\Phi = \text{ME-lim}\,\Phi_\tau$ if and only if, for all $x \in \mathrm{H}$, there exists $(x_\tau)_{\tau>0} \to x$ such that $\limsup \Phi_\tau(x_\tau) \leq \Phi(x)$ and, for all $(x_\tau)_{\tau>0} \rightharpoonup x$, $\liminf \Phi_\tau(x_\tau) \geq \Phi(x)$.*

2.2 Some basics of convex analysis

The *domain* and the *graph* of a set-valued map $A : \mathrm{H} \rightrightarrows \mathrm{H}$ are respectively given by

$$\mathrm{D}(A) := \{x \in \mathrm{H} \mid A(x) \neq \emptyset\} \quad \text{and} \quad \mathrm{Gr}(A) := \{(x,y) \in \mathrm{H} \times \mathrm{H} \mid y \in A(x)\}.$$

We denote by $A^{-1} : \mathrm{H} \rightrightarrows \mathrm{H}$ the set-valued map defined by

$$A^{-1}(y) := \{x \in \mathrm{H} \mid y \in A(x)\},$$

for all $y \in \mathrm{H}$. For all x, $y \in \mathrm{H}$, note that $y \in A(x)$ if and only if $x \in A^{-1}(y)$. The *range* of A is given by

$$\mathrm{R}(A) := \{y \in \mathrm{H} \mid A^{-1}(y) \neq \emptyset\} = \mathrm{D}(A^{-1}).$$

Let A, $B : \mathrm{H} \rightrightarrows \mathrm{H}$ be two set-valued maps. The sum $A + B : \mathrm{H} \rightrightarrows \mathrm{H}$ is defined by

$$(A+B)(x) := \{y_A + y_B \mid y_A \in A(x),\ y_B \in B(x)\},$$

for all $x \in \mathrm{H}$. Finally, a set-valued map $A : \mathrm{H} \rightrightarrows \mathrm{H}$ is said to be *single-valued* if $A(x)$ is a singleton for all $x \in \mathrm{H}$. In that case, it holds in particular that $\mathrm{D}(A) = \mathrm{H}$ and we write $A : \mathrm{H} \to \mathrm{H}$ (instead of $A : \mathrm{H} \rightrightarrows \mathrm{H}$, by identifying A to a standard map).

A set-valued map $A : \mathrm{H} \rightrightarrows \mathrm{H}$ is said to be *monotone* if

$$\forall (x_1, y_1), (x_2, y_2) \in \mathrm{Gr}(A), \quad \langle y_2 - y_1, x_2 - x_1 \rangle_{\mathrm{H}} \geq 0.$$

Moreover A is said to be *maximal monotone* if $\mathrm{Gr}(A) \subset \mathrm{Gr}(B)$ for some monotone set-valued map $B : \mathrm{H} \rightrightarrows \mathrm{H}$ implies that $A = B$. From Minty's theorem (see, e.g., [23]), it is well-known that a monotone operator $A : \mathrm{H} \rightrightarrows \mathrm{H}$ is maximal if and only if $\mathrm{R}(\mathrm{Id} + A) = \mathrm{H}$.

In what follows, as usual in the literature, we denote by $\Gamma_0(\mathrm{H})$ the set of all extended real-valued functions $\Phi : \mathrm{H} \to \mathbb{R} \cup \{+\infty\}$ with a nonempty closed convex epigraph. Let $\Phi \in \Gamma_0(\mathrm{H})$. We denote by $\partial\Phi : \mathrm{H} \rightrightarrows \mathrm{H}$ the *subdifferential operator* of Φ defined by

$$\partial\Phi(x) := \{y \in \mathrm{H} \mid \forall z \in \mathrm{H},\ \langle y, z - x \rangle_{\mathrm{H}} \leq \Phi(z) - \Phi(x)\},$$

for all $x \in \mathrm{H}$. Moreover we denote by $\mathrm{prox}_\Phi : \mathrm{H} \rightrightarrows \mathrm{H}$ the *proximal operator* (also well-known as *proximity operator*) of Φ defined by

$$\mathrm{prox}_\Phi := (\mathrm{Id} + \partial\Phi)^{-1}.$$

Recall that $\partial\Phi$ is a maximal monotone operator (see, e.g., [27] or [30, Theorem 12.17 p.542] for details). As a consequence it can be easily deduced that $\mathrm{prox}_\Phi : \mathrm{H} \to \mathrm{H}$ is a single-valued map.

We conclude this section by noting that if $\Phi = \mathrm{I}_\mathrm{K}$ is the *indicator function* of a nonempty closed convex subset K of H, that is, $\Phi(x) = 0$ if $x \in \mathrm{K}$, and $\Phi(x) = +\infty$ otherwise, then $\Phi \in \Gamma_0(\mathrm{H})$ and $\partial\Phi = \mathrm{N}_\mathrm{K}$ coincides with the classical *normal cone* of K (in the sense of convex analysis) and $\mathrm{prox}_\Phi = \mathrm{proj}_\mathrm{K}$ coincides with the classical *projection operator* onto K.

2.3 Twice epi-differentiability

For a given $\Phi \in \Gamma_0(\mathrm{H})$, we define the following second-order difference quotient functions by

$$\begin{array}{rccl} \Delta_\tau^2\Phi(x|y): & \mathrm{H} & \longrightarrow & \mathbb{R} \cup \{+\infty\} \\ & z & \longmapsto & \Delta_\tau^2\Phi(x|y)(z) := \dfrac{\Phi(x+\tau z) - \Phi(x) - \tau\langle y, z\rangle_\mathrm{H}}{\tau^2}, \end{array}$$

for all $\tau > 0$, $x \in \mathrm{dom}(\Phi)$ and $y \in \partial\Phi(x)$.

Remark 2.4. R.T. Rockafellar defined originally in [29] the second-order difference quotient functions with a factor $\frac{1}{2}$ in the denominator. The main reason to include this factor is getting the second-order epi-derivatives agree with classical second-order derivatives in the case where both exist. Since there is no confusion in the present work, and for simplicity, we omit the factor $\frac{1}{2}$ in our definition.

Definition 2.5 (Twice epi-differentiability)**.** Let $\Phi \in \Gamma_0(\mathrm{H})$. We say that Φ is *twice epi-differentiable* at $x \in \mathrm{dom}(\Phi)$ for $y \in \partial\Phi(x)$ if $(\Delta_\tau^2\Phi(x|y))_{\tau>0}$ Mosco epi-converges. In that case we denote by

$$\mathrm{d}_e^2\Phi(x|y) := \text{ME-lim}\, \Delta_\tau^2\Phi(x|y),$$

which is called the *second-order epi-derivative* of Φ at x for y.

Example 2.6. Let $|\cdot| : \mathbb{R} \to \mathbb{R}$ stand for the standard absolute value map. It is clear that $|\cdot| \in \Gamma_0(\mathbb{R})$ with

$$\partial\,|\cdot|\,(x) = \begin{cases} \{-1\} & \text{if} \quad x < 0, \\ [-1,1] & \text{if} \quad x = 0, \\ \{1\} & \text{if} \quad x > 0, \end{cases}$$

for all $x \in \mathbb{R}$. One can easily see that $|\cdot|$ is twice epi-differentiable at any $x \in \mathbb{R}$ for all $y \in \partial\,|\cdot|\,(x)$ with $\mathrm{d}_e^2\,|\cdot|\,(x|y) = \mathrm{I}_{\mathrm{K}_{x,y}}$ where

$$\mathrm{K}_{x,y} := \begin{cases} \mathbb{R} & \text{if} \quad x \neq 0, \\ \mathbb{R}_- & \text{if} \quad x = 0 \text{ and } y = -1, \\ \{0\} & \text{if} \quad x = 0 \text{ and } y \in (-1,1), \\ \mathbb{R}_+ & \text{if} \quad x = 0 \text{ and } y = 1, \end{cases}$$

is a nonempty closed convex subset of $\mathbb{R}$. In particular we have $\mathrm{d}_e^2\,|\cdot|\,(x|y) \in \Gamma_0(\mathbb{R})$ with

$$\mathrm{prox}_{\mathrm{d}_e^2|\cdot|(x|y)} = \mathrm{proj}_{\mathrm{K}_{x,y}},$$

for all $x \in \mathbb{R}$ and $y \in \partial\,|\cdot|\,(x)$.

We conclude this section by recalling two propositions (see, e.g., [29, 30] for the finite-dimensional case and [1, 9] for the infinite-dimensional one). We bring to the attention of the reader that Formula (1) given in Proposition 2.8 is the key point in order to derive our main result (Theorem 3.16) in the next section.

Proposition 2.7. *Let $\Phi \in \Gamma_0(\mathrm{H})$. If Φ is twice epi-differentiable at $x \in \mathrm{dom}(\Phi)$ for $y \in \partial\Phi(x)$, then $\mathrm{d}_e^2\Phi(x|y) \in \Gamma_0(\mathrm{H})$.*

Proposition 2.8. *Let $\Phi \in \Gamma_0(\mathrm{H})$ and $F : \mathbb{R}_+ \to \mathrm{H}$ be a given function. We consider the function $u : \mathbb{R}_+ \to \mathrm{H}$ defined by $u(t) := \mathrm{prox}_\Phi(F(t))$ for all $t \geq 0$. If the conditions*

(i) *F is differentiable at $t = 0$;*

(ii) *Φ is twice epi-differentiable at $u(0)$ for $F(0) - u(0)$;*

are both satisfied, then u is differentiable at $t = 0$ with

$$u'(0) = \mathrm{prox}_{\mathrm{d}_e^2\Phi(u(0)|F(0)-u(0))}(F'(0)). \tag{1}$$

3 Main result

This section is dedicated to the main result (Theorem 3.16) of the present paper. As a first step we provide in Section 3.1 the functional settings and recall some basic results. A general scalar Signorini-type problem is presented and investigated in Section 3.2. A general scalar Tresca-type problem is introduced and studied in Section 3.3. On this occasion the subdifferential of the corresponding Tresca-type functional is characterized. Finally, in Section 3.4, we establish our main result (Theorem 3.16) which claims, roughly speaking, that the derivative of a parameterized Tresca-type problem satisfies Signorini's conditions.

3.1 Functional setting and basic results

In the whole section we fix $d \in \mathbb{N}^*$ being a positive integer. Let $\Omega \subset \mathbb{R}^d$ be a nonempty bounded connected open subset of $\mathbb{R}^d$ with a Lipschitz continuous boundary $\Gamma := \partial\Omega$. In what follows $\mathrm{C}_\mathrm{c}^\infty(\Omega)$ stands for the standard space of infinitely differentiable real functions defined and compactly supported on Ω, and $\mathcal{D}'(\Omega)$ stands for the corresponding classical distributions space. Then we denote by $\mathrm{L}^2(\Omega)$, $\mathrm{L}^2(\Gamma)$, $\mathrm{L}^1(\Gamma)$, $\mathrm{H}^1(\Omega)$, $\mathrm{H}^{1/2}(\Gamma)$, $\mathrm{H}^{-1/2}(\Gamma)$, etc., the usual Lebesgue and Sobolev spaces endowed with their standard norms. We recall that the continuous and dense embeddings $\mathrm{H}^1(\Omega) \hookrightarrow \mathrm{L}^2(\Omega)$, $\mathrm{H}^1(\Omega) \hookrightarrow \mathrm{H}^{1/2}(\Gamma) \hookrightarrow \mathrm{L}^2(\Gamma) \hookrightarrow \mathrm{H}^{-1/2}(\Gamma)$ and $\mathrm{L}^2(\Gamma) \hookrightarrow \mathrm{L}^1(\Gamma)$ hold. We also recall that the continuous and dense embedding $\mathrm{H}^1(\Omega) \hookrightarrow \mathrm{L}^2(\Gamma)$ is compact. We refer for instance to the standard books [7, 11]. Finally we denote by $\mathrm{B}_\Gamma(s,\varepsilon) \subset \Gamma$ the usual open ball of Γ centered at some $s \in \Gamma$ with some radius $\varepsilon > 0$.

Proposition 3.1 (Green formula). *Let $w \in \mathrm{H}^1(\Omega)$. If $\Delta w \in \mathrm{L}^2(\Omega)$, then ∇w admits a normal trace $\partial_\mathrm{n} w \in \mathrm{H}^{-1/2}(\Gamma)$ on Γ, and the Green formula*

$$\int_\Omega \Delta w \, \varphi + \int_\Omega \nabla w \cdot \nabla\varphi = \langle \partial_\mathrm{n} w, \varphi \rangle_{\mathrm{H}^{-1/2}(\Gamma)\times\mathrm{H}^{1/2}(\Gamma)},$$

holds true for all $\varphi \in \mathrm{H}^1(\Omega)$*. Moreover* $\partial_{\mathrm{n}} w$ *can be identified to an element of* $\mathrm{L}^2(\Gamma)$ *with*

$$\langle \partial_{\mathrm{n}} w, \varphi \rangle_{\mathrm{H}^{-1/2}(\Gamma)\times\mathrm{H}^{1/2}(\Gamma)} = \int_{\Gamma} \partial_{\mathrm{n}} w \, \varphi,$$

for all $\varphi \in \mathrm{H}^{1/2}(\Gamma)$*, if and only if there exists* $c \geq 0$ *such that*

$$\left| \langle \partial_{\mathrm{n}} w, \varphi \rangle_{\mathrm{H}^{-1/2}(\Gamma)\times\mathrm{H}^{1/2}(\Gamma)} \right| \leq c \|\varphi\|_{\mathrm{L}^2(\Gamma)},$$

for all $\varphi \in \mathrm{H}^{1/2}(\Gamma)$*.*

Proof. We refer to [13, Corollary 2.6 p.28] for the first part of Proposition 3.1. The second part directly follows from Proposition A.1 in Appendix A. □

Let us fix some function $f \in \mathrm{L}^2(\Omega)$. In what follows we will consider the classical Neumann problem given by

$$\begin{cases} -\Delta F + F = f & \text{in } \Omega, \\ \partial_{\mathrm{n}} F = 0 & \text{on } \Gamma. \end{cases} \tag{NP}$$

A solution to Problem (NP) is a function $F \in \mathrm{H}^1(\Omega)$ which satisfies $-\Delta F + F = f$ in $\mathcal{D}'(\Omega)$ and such that $\partial_{\mathrm{n}} F \in \mathrm{L}^2(\Gamma)$ with $\partial_{\mathrm{n}} F(s) = 0$ for almost every $s \in \Gamma$. Let us recall the very classical variational formulation and the well-posedness of Problem (NP) in the next propositions. We refer, among others, to the standard books [7, 11].

Proposition 3.2. *A function* $F \in \mathrm{H}^1(\Omega)$ *is a solution to Problem* (NP) *if and only if it satisfies the variational equality given by*

$$\int_{\Omega} \nabla F \cdot \nabla \varphi + \int_{\Omega} F\varphi = \int_{\Omega} f\varphi,$$

for all $\varphi \in \mathrm{H}^1(\Omega)$*.*

Proposition 3.3. *Problem* (NP) *admits a unique solution* $F \in \mathrm{H}^1(\Omega)$*. Moreover it holds that* $\|F\|_{\mathrm{H}^1(\Omega)} \leq \|f\|_{\mathrm{L}^2(\Omega)}$*.*

3.2 A general scalar Signorini-type problem

In the whole section we consider four (possibly empty) measurable pairwise disjoint subsets Γ_{N}, Γ_{D}, $\Gamma_{\mathrm{S}-}$, $\Gamma_{\mathrm{S}+}$ of Γ such that the decomposition

$$\Gamma = \Gamma_{\mathrm{N}} \cup \Gamma_{\mathrm{D}} \cup \Gamma_{\mathrm{S}-} \cup \Gamma_{\mathrm{S}+},$$

holds true. The general scalar Signorini-type problem considered in this paper has the following form

$$\begin{cases} -\Delta u + u = f & \text{in } \Omega, \\ \partial_{\mathrm{n}} u = 0 & \text{on } \Gamma_{\mathrm{N}}, \\ u = 0 & \text{on } \Gamma_{\mathrm{D}}, \\ u \leq 0,\ \partial_{\mathrm{n}} u \leq 0 \text{ and } u\partial_{\mathrm{n}} u = 0 & \text{on } \Gamma_{\mathrm{S}-}, \\ u \geq 0,\ \partial_{\mathrm{n}} u \geq 0 \text{ and } u\partial_{\mathrm{n}} u = 0 & \text{on } \Gamma_{\mathrm{S}+}. \end{cases} \tag{SP}$$

Note that similar scalar versions of the usual (vectorial) Signorini problem can be found in the literature (see, e.g., [22, Section 1]).

Definition 3.4. Let $u \in \mathrm{H}^1(\Omega)$.

(i) The function u is said to be a *(strong) solution* to Problem (SP) if it satisfies $-\Delta u + u = f$ in $\mathcal{D}'(\Omega)$ and $\partial_{\mathrm{n}} u \in \mathrm{L}^2(\Gamma)$ with the four boundary conditions being satisfied almost everywhere on Γ.

(ii) The function u is said to be a *weak solution* to Problem (SP) if $u \in \mathcal{K}^1$ and u satisfies the variational inequality given by

$$\int_\Omega \nabla u \cdot \nabla(\varphi - u) + \int_\Omega u(\varphi - u) \geq \int_\Omega f(\varphi - u),$$

for all $\varphi \in \mathcal{K}^1$, where $\mathcal{K}^1$ is the nonempty closed convex subset of $\mathrm{H}^1(\Omega)$ given by

$$\mathcal{K}^1 := \{\varphi \in \mathrm{H}^1(\Omega) \mid \varphi \leq 0 \text{ on } \Gamma_{\mathrm{S}-},\ \varphi = 0 \text{ on } \Gamma_{\mathrm{D}} \text{ and } \varphi \geq 0 \text{ on } \Gamma_{\mathrm{S}+}\}.$$

Remark 3.5. We introduce in Definition 3.4 two different concepts of solutions to Problem (SP). In fact, without additional assumptions, we are only able to prove the existence and uniqueness of a weak solution (see Proposition 3.6 below). Nevertheless we provide in Proposition 3.8 a sufficient condition which ensures that a weak solution to Problem (SP) is a solution in the strong sense.

Proposition 3.6. *Problem* (SP) *admits a unique weak solution given by*

$$u = \mathrm{proj}_{\mathcal{K}^1}(F),$$

where $F \in \mathrm{H}^1(\Omega)$ *is the unique solution to Problem* (NP).

Proof. Let $u \in \mathrm{H}^1(\Omega)$. From Definition 3.4 and Proposition 3.2 we know that u is a weak solution to Problem (SP) if and only if $u \in \mathcal{K}^1$ and $\langle F - u, \varphi - u \rangle_{\mathrm{H}^1(\Omega)} \leq 0$ for all $\varphi \in \mathcal{K}^1$, that is exactly, if and only if $u = \mathrm{proj}_{\mathcal{K}^1}(F)$. □

Definition 3.7. The decomposition $\Gamma = \Gamma_{\mathrm{N}} \cup \Gamma_{\mathrm{D}} \cup \Gamma_{\mathrm{S}-} \cup \Gamma_{\mathrm{S}+}$ is said to be *consistent* if the following conditions are both fulfilled:

(i) Almost every point of Γ is an interior point of one of the subsets Γ_{N}, Γ_{D}, $\Gamma_{\mathrm{S}-}$ and $\Gamma_{\mathrm{S}+}$;

(ii) The nonempty closed convex subset $\mathcal{K}^{1/2}$ of $\mathrm{H}^{1/2}(\Gamma)$ defined by

$$\mathcal{K}^{1/2} := \{\varphi \in \mathrm{H}^{1/2}(\Gamma) \mid \varphi \leq 0 \text{ on } \Gamma_{\mathrm{S}-},\ \varphi = 0 \text{ on } \Gamma_{\mathrm{D}} \text{ and } \varphi \geq 0 \text{ on } \Gamma_{\mathrm{S}+}\},$$

is dense in the nonempty closed convex subset $\mathcal{K}^0$ of $\mathrm{L}^2(\Gamma)$ defined by

$$\mathcal{K}^0 := \{\varphi \in \mathrm{L}^2(\Gamma) \mid \varphi \leq 0 \text{ on } \Gamma_{\mathrm{S}-},\ \varphi = 0 \text{ on } \Gamma_{\mathrm{D}} \text{ and } \varphi \geq 0 \text{ on } \Gamma_{\mathrm{S}+}\}.$$

Proposition 3.8. *Let* $u \in \mathrm{H}^1(\Omega)$. *Then:*

(i) *If u is a (strong) solution to Problem* (SP)*, then u is a weak solution to Problem* (SP)*.*

(ii) *If u is a weak solution to Problem* (SP) *with* $\partial_{\mathrm{n}} u \in \mathrm{L}^2(\Gamma)$ *and the decomposition* $\Gamma = \Gamma_{\mathrm{N}} \cup \Gamma_{\mathrm{D}} \cup \Gamma_{\mathrm{S}-} \cup \Gamma_{\mathrm{S}+}$ *is consistent, then u is a (strong) solution to Problem* (SP)*.*

Proof. (i) Assume that u is a (strong) solution to Problem (SP). From the boundary conditions, note that $u \in \mathcal{K}^1$. Moreover it holds that $\Delta u = u - f \in \mathrm{L}^2(\Omega)$. Since moreover $\partial_{\mathrm{n}} u \in \mathrm{L}^2(\Gamma)$, the Green formula leads to

$$\int_\Omega \nabla u \cdot \nabla(\varphi - u) + \int_\Omega u(\varphi - u) = \int_\Omega f(\varphi - u) + \int_\Gamma \partial_{\mathrm{n}} u(\varphi - u) \geq \int_\Omega f(\varphi - u),$$

for all $\varphi \in \mathcal{K}^1$ (the last inequality coming from the boundary conditions of u and φ). This concludes the proof of the first item. (ii) Assume that u is a weak solution to Problem (SP) with $\partial_{\mathrm{n}} u \in \mathrm{L}^2(\Gamma)$ and that the decomposition $\Gamma = \Gamma_{\mathrm{N}} \cup \Gamma_{\mathrm{D}} \cup \Gamma_{\mathrm{S}-} \cup \Gamma_{\mathrm{S}+}$ is consistent. In particular we have $u \in \mathcal{K}^1$. Considering the test functions $\varphi = u \pm \psi \in \mathcal{K}^1$ with $\psi \in \mathrm{C}^\infty_{\mathrm{c}}(\Omega)$, we get that $-\Delta u + u = f$ in $\mathcal{D}'(\Omega)$ and thus $\Delta u = u - f \in \mathrm{L}^2(\Omega)$. Since $\partial_{\mathrm{n}} u \in \mathrm{L}^2(\Gamma)$, the Green formula leads to

$$\int_\Gamma \partial_{\mathrm{n}} u(\varphi - u) \geq 0,$$

for all $\varphi \in \mathcal{K}^1$, and thus for all $\varphi \in \mathcal{K}^{1/2}$. From density of $\mathcal{K}^{1/2}$ in $\mathcal{K}^0$, the above inequality is satisfied for all $\varphi \in \mathcal{K}^0$. Let $s \in \Gamma$ be a Lebesgue point of $\partial_{\mathrm{n}} u \in \mathrm{L}^2(\Gamma)$ which is moreover in the interior of one of the subsets Γ_{N}, Γ_{D}, $\Gamma_{\mathrm{S}-}$ and $\Gamma_{\mathrm{S}+}$. If $s \in \Gamma_{\mathrm{N}}$, we consider the test functions $\varphi = u \pm \psi \in \mathcal{K}^0$ where

$$\psi := \begin{cases} 1 & \text{on } \mathrm{B}_\Gamma(s, \varepsilon), \\ 0 & \text{on } \Gamma \backslash \mathrm{B}_\Gamma(s, \varepsilon), \end{cases}$$

for small enough $\varepsilon > 0$ satisfying $\mathrm{B}_\Gamma(s, \varepsilon) \subset \Gamma_{\mathrm{N}}$. We get that

$$\frac{\pm 1}{|\mathrm{B}_\Gamma(s, \varepsilon)|} \int_{\mathrm{B}_\Gamma(s,\varepsilon)} \partial_{\mathrm{n}} u \geq 0.$$

Taking the limit $\varepsilon \to 0^+$, we obtain that $\partial_{\mathrm{n}} u(s) = 0$. Adapting appropriately the above strategy to the case $s \in \Gamma_{\mathrm{S}-}$ with the test function $\varphi = u - \psi \in \mathcal{K}^0$ (resp. $s \in \Gamma_{\mathrm{S}+}$ with the test function $\varphi = u + \psi \in \mathcal{K}^0$), we get that $\partial_{\mathrm{n}} u(s) \leq 0$ (resp. $\partial_{\mathrm{n}} u(s) \geq 0$). Finally considering the test functions $\varphi = 0 \in \mathcal{K}^0$ and $\varphi = 2u \in \mathcal{K}^0$, we get that

$$\int_\Gamma u \partial_{\mathrm{n}} u = 0,$$

while the integrand is nonnegative almost everywhere on Γ from the previous assertions. We conclude that $u(s)\partial_{\mathrm{n}} u(s) = 0$ for almost every $s \in \Gamma_{\mathrm{S}-}$ and almost every $s \in \Gamma_{\mathrm{S}+}$. The proof of (ii) is thereby completed. □

Remark 3.9. The notion of *consistent decomposition* introduced in Definition 3.7 does not play any crucial role in our study. Indeed our aim is not to establish a minimal sufficient condition which guarantees the equivalence of the two notions of strong and weak solutions to Problem (SP). Nevertheless one can easily note that the notion of consistent decomposition is quite unrestrictive.

3.3 A general scalar Tresca-type problem and the corresponding Tresca-type functional

The general scalar Tresca-type problem considered in this paper has the form

$$\begin{cases} -\Delta u + u = f & \text{in } \Omega, \\ |\partial_{\mathrm{n}} u| \leq 1 \text{ and } u \partial_{\mathrm{n}} u = -|u| & \text{on } \Gamma. \end{cases} \tag{TP}$$

A solution to Problem (TP) is a function $u \in \mathrm{H}^1(\Omega)$ which satisfies $-\Delta u + u = f$ in $\mathcal{D}'(\Omega)$ and such that $\partial_{\mathrm{n}} u \in \mathrm{L}^2(\Gamma)$ with $|\partial_{\mathrm{n}} u(s)| \leq 1$ and $u(s)\partial_{\mathrm{n}} u(s) = -|u(s)|$ for almost every $s \in \Gamma$.

Remark 3.10. If $u \in \mathrm{H}^1(\Omega)$ is a solution to Problem (TP), note that the boundary conditions impose some restrictions on the boundary values of u and $\partial_{\mathrm{n}} u$. Firstly, if $u(s) \neq 0$ for some $s \in \Gamma$, then $\partial_{\mathrm{n}} u(s) = -\mathrm{sign}(u(s))$. Secondly, if $\partial_{\mathrm{n}} u(s) \in (-1, 1)$ for some $s \in \Gamma$, then $u(s) = 0$. Finally, if $\partial_{\mathrm{n}} u(s) = -1$ (resp. $\partial_{\mathrm{n}} u(s) = 1$) for some $s \in \Gamma$, then $u(s) \geq 0$ (resp. $u(s) \leq 0$).

Proposition 3.11. *A function $u \in \mathrm{H}^1(\Omega)$ is a solution to Problem* (TP) *if and only if it satisfies the variational inequality given by*

$$\int_\Omega \nabla u \cdot \nabla(\varphi - u) + \int_\Omega u(\varphi - u) + \int_\Gamma |\varphi| - \int_\Gamma |u| \geq \int_\Omega f(\varphi - u),$$

for all $\varphi \in \mathrm{H}^1(\Omega)$.

Proof. Firstly let $u \in \mathrm{H}^1(\Omega)$ be a solution to Problem (TP). It holds that $\Delta u = u - f \in \mathrm{L}^2(\Omega)$. Since moreover $\partial_{\mathrm{n}} u \in \mathrm{L}^2(\Gamma)$, the Green formula leads to

$$\int_\Omega \nabla u \cdot \nabla(\varphi - u) + \int_\Omega u(\varphi - u) - \int_\Gamma \partial_{\mathrm{n}} u(\varphi - u) = \int_\Omega f(\varphi - u),$$

for all $\varphi \in \mathrm{H}^1(\Omega)$. Separating the three cases $u(s) = 0$, $u(s) > 0$ (with $\partial_{\mathrm{n}} u(s) = -1$) and $u(s) < 0$ (with $\partial_{\mathrm{n}} u(s) = 1$), one can easily prove that $-\partial_{\mathrm{n}} u(s)(\varphi(s) - u(s)) \leq |\varphi(s)| - |u(s)|$ for almost every $s \in \Gamma$ and all $\varphi \in \mathrm{H}^1(\Omega)$. This concludes the first part of the proof. Conversely let $u \in \mathrm{H}^1(\Omega)$ satisfying the variational inequality. Considering the test functions $\varphi = u \pm \psi \in \mathrm{H}^1(\Omega)$ with $\psi \in \mathrm{C}^\infty_c(\Omega)$, we get that $-\Delta u + u = f$ in $\mathcal{D}'(\Omega)$ and thus we obtain $\Delta u = u - f \in \mathrm{L}^2(\Omega)$. The Green formula leads to

$$-\langle \partial_{\mathrm{n}} u, \varphi - u \rangle_{\mathrm{H}^{-1/2}(\Gamma) \times \mathrm{H}^{1/2}(\Gamma)} \leq \int_\Gamma |\varphi| - \int_\Gamma |u|,$$

for all $\varphi \in \mathrm{H}^1(\Omega)$. Considering the test functions $\varphi = u \pm \psi \in \mathrm{H}^1(\Omega)$ with $\psi \in \mathrm{H}^1(\Omega)$ and using the continuous embedding $\mathrm{L}^2(\Gamma) \hookrightarrow \mathrm{L}^1(\Gamma)$, there exists $c \geq 0$ such that

$$|\langle \partial_{\mathrm{n}} u, \psi \rangle_{\mathrm{H}^{-1/2}(\Gamma) \times \mathrm{H}^{1/2}(\Gamma)}| \leq c \|\psi\|_{\mathrm{L}^2(\Gamma)},$$

for all $\psi \in \mathrm{H}^{1/2}(\Gamma)$. We deduce from Proposition 3.1 that $\partial_{\mathrm{n}} u \in \mathrm{L}^2(\Gamma)$ and that

$$-\int_\Gamma \partial_{\mathrm{n}} u(\varphi - u) \leq \int_\Gamma |\varphi| - \int_\Gamma |u|,$$

for all $\varphi \in \mathrm{H}^1(\Omega)$, and thus for all $\varphi \in \mathrm{L}^2(\Gamma)$ from the density of $\mathrm{H}^{1/2}(\Gamma)$ in $\mathrm{L}^2(\Gamma)$ and the continuous embedding $\mathrm{L}^2(\Gamma) \hookrightarrow \mathrm{L}^1(\Gamma)$. Let $s \in \Gamma$ be a Lebesgue point of $\partial_{\mathrm{n}} u \in \mathrm{L}^2(\Gamma)$ and let us consider the test functions $\varphi = u \pm \psi \in \mathrm{L}^2(\Gamma)$ where $\psi \in \mathrm{L}^2(\Gamma)$ is defined by

$$\psi := \begin{cases} 1 & \text{on } \mathrm{B}_\Gamma(s, \varepsilon), \\ 0 & \text{on } \Gamma \backslash \mathrm{B}_\Gamma(s, \varepsilon), \end{cases}$$

with $\varepsilon > 0$. We deduce that

$$\frac{\pm 1}{|\mathrm{B}_\Gamma(s, \varepsilon)|} \int_{\mathrm{B}_\Gamma(s,\varepsilon)} \partial_{\mathrm{n}} u \leq 1.$$

Taking the limit $\varepsilon \to 0^+$, we obtain that $|\partial_{\mathrm{n}} u(s)| \leq 1$, and thus $u(s)\partial_{\mathrm{n}} u(s) + |u(s)| \geq 0$. Moreover, by considering the test function $\varphi = 0 \in \mathrm{L}^2(\Gamma)$, we get that

$$\int_\Gamma u\partial_{\mathrm{n}} u + |u| \leq 0,$$

while the integrand is nonnegative almost everywhere on Γ from the previous assertions. The proof is complete. $\square$

Proposition 3.12. *Problem* (TP) *admits a unique solution given by*

$$u = \mathrm{prox}_\Phi(F),$$

where $F \in \mathrm{H}^1(\Omega)$ is the unique solution to Problem (NP) *and $\Phi \in \Gamma_0(\mathrm{H}^1(\Omega))$ is the Tresca-type functional defined by*

$$\begin{array}{rccl} \Phi : & \mathrm{H}^1(\Omega) & \longrightarrow & \mathbb{R} \\ & w & \longmapsto & \Phi(w) := \displaystyle\int_\Gamma |w| \, . \end{array}$$

Proof. Firstly note that the Tresca-type functional Φ belongs to $\Gamma_0(\mathrm{H}^1(\Omega))$, in particular thanks to the continuous embedding $\mathrm{H}^1(\Omega) \hookrightarrow \mathrm{L}^2(\Gamma)$. Now let $u \in \mathrm{H}^1(\Omega)$. From Propositions 3.2 and 3.11, we know that u is a solution to Problem (TP) if and only if $\langle F - u, \varphi - u \rangle_{\mathrm{H}^1(\Omega)} \leq \Phi(\varphi) - \Phi(u)$ for all $\varphi \in \mathrm{H}^1(\Omega)$, that is exactly, if and only if $F - u \in \partial\Phi(u)$, that is, if and only if $u = \mathrm{prox}_\Phi(F)$. The proof is complete. $\square$

For the needs of our main result, we state some preliminary results on the Tresca-type functional. To this aim we introduce the Auxiliary Problem

$$\left\{ \begin{array}{rl} -\Delta v + v = 0 & \text{in } \Omega, \\ \partial_{\mathrm{n}} v(s) \in \partial \left|\cdot\right| (u(s)) & \text{on } \Gamma, \end{array} \right. \tag{AP$_u$}$$

for all $u \in \mathrm{H}^1(\Omega)$. A solution to Problem (AP$_u$) for some $u \in \mathrm{H}^1(\Omega)$ is a function $v \in \mathrm{H}^1(\Omega)$ which satisfies $-\Delta v + v = 0$ in $\mathcal{D}'(\Omega)$ and such that $\partial_{\mathrm{n}} v \in \mathrm{L}^2(\Gamma)$ with $\partial_{\mathrm{n}} v(s) \in \partial \left|\cdot\right| (u(s))$ for almost every $s \in \Gamma$.

Lemma 3.13. *It holds that*

$$\partial\Phi(u) = \textit{the set of solutions to Problem } (\mathrm{AP}_u),$$

for all $u \in \mathrm{H}^1(\Omega)$.

Proof. Let $u \in \mathrm{H}^1(\Omega)$ and let us prove the two inclusions separately. Firstly let $v \in \mathrm{H}^1(\Omega)$ be a solution to Problem (AP$_u$) and let us prove that $v \in \partial\Phi(u)$. Since $\partial_{\mathrm{n}} v(s) \in \partial \left|\cdot\right| (u(s))$, we get that $\partial_{\mathrm{n}} v(s)(\varphi(s) - u(s)) \leq |\varphi(s)| - |u(s)|$ for almost every $s \in \Gamma$ and for all $\varphi \in \mathrm{H}^1(\Omega)$. Since $\partial_{\mathrm{n}} v \in \mathrm{L}^2(\Gamma)$, we get that

$$\int_\Gamma \partial_{\mathrm{n}} v(\varphi - u) \leq \int_\Gamma |\varphi| - \int_\Gamma |u| \, ,$$

for all $\varphi \in \mathrm{H}^1(\Omega)$. Since $-\Delta v + v = 0$ in $\mathcal{D}'(\Omega)$ and thus $\Delta v = v \in \mathrm{L}^2(\Omega)$, the Green formula leads to

$$\langle v, \varphi - u \rangle_{\mathrm{H}^1(\Omega)} \leq \int_\Gamma |\varphi| - \int_\Gamma |u| \, ,$$

for all $\varphi \in \mathrm{H}^1(\Omega)$, which means that $v \in \partial\Phi(u)$. The proof of the first inclusion is complete. Conversely let $v \in \partial\Phi(u)$ and let us prove that v is a solution to Problem (AP_u). Since $v \in \partial\Phi(u)$ it holds that

$$\int_\Omega \nabla v \cdot \nabla(\varphi - u) + \int_\Omega v(\varphi - u) \leq \int_\Gamma |\varphi| - \int_\Gamma |u|,$$

for all $\varphi \in \mathrm{H}^1(\Omega)$. Considering the test functions $\varphi = u \pm \psi \in \mathrm{H}^1(\Omega)$ with $\psi \in \mathrm{C}_c^\infty(\Omega)$, we get that $-\Delta v + v = 0$ in $\mathcal{D}'(\Omega)$ and thus $\Delta v = v \in \mathrm{L}^2(\Omega)$. The Green formula leads to

$$\langle \partial_\mathrm{n} v, \varphi - u \rangle_{\mathrm{H}^{-1/2}(\Gamma)\times\mathrm{H}^{1/2}(\Gamma)} \leq \int_\Gamma |\varphi| - \int_\Gamma |u|,$$

for all $\varphi \in \mathrm{H}^1(\Omega)$. Using the test functions $\varphi = u \pm \psi \in \mathrm{H}^1(\Omega)$ with $\psi \in \mathrm{H}^1(\Omega)$ and the continuous embedding $\mathrm{L}^2(\Gamma) \hookrightarrow \mathrm{L}^1(\Gamma)$, we deduce that there exists a constant $c \geq 0$ such that

$$|\langle \partial_\mathrm{n} v, \psi \rangle_{\mathrm{H}^{-1/2}(\Gamma)\times\mathrm{H}^{1/2}(\Gamma)}| \leq c \, \|\psi\|_{\mathrm{L}^2(\Gamma)},$$

for all $\psi \in \mathrm{H}^{1/2}(\Gamma)$. From Proposition 3.1, we deduce that $\partial_\mathrm{n} v \in \mathrm{L}^2(\Gamma)$ and, from density of $\mathrm{H}^{1/2}(\Gamma)$ in $\mathrm{L}^2(\Gamma)$, that

$$\int_\Gamma \partial_\mathrm{n} v(\varphi - u) \leq \int_\Gamma |\varphi| - \int_\Gamma |u|,$$

for all $\varphi \in \mathrm{L}^2(\Gamma)$. Let $s \in \Gamma$ be a Lebesgue point of $\partial_\mathrm{n} v \in \mathrm{L}^2(\Gamma)$, of the product $u\partial_\mathrm{n} v \in \mathrm{L}^1(\Gamma)$ and of $|u| \in \mathrm{L}^2(\Gamma)$. Using the test function $\varphi \in \mathrm{L}^2(\Gamma)$ given by

$$\varphi := \begin{cases} \xi & \text{on } \mathrm{B}_\Gamma(s,\varepsilon), \\ u & \text{on } \Gamma\backslash\mathrm{B}_\Gamma(s,\varepsilon), \end{cases}$$

where $\varepsilon > 0$ and $\xi \in \mathbb{R}$, we obtain that

$$\frac{1}{|\mathrm{B}_\Gamma(s,\varepsilon)|}\int_{\mathrm{B}_\Gamma(s,\varepsilon)} \partial_\mathrm{n} v(\xi - u) \leq \frac{1}{|\mathrm{B}_\Gamma(s,\varepsilon)|}\int_{\mathrm{B}_\Gamma(s,\varepsilon)} |\xi| - \frac{1}{|\mathrm{B}_\Gamma(s,\varepsilon)|}\int_{\mathrm{B}_\Gamma(s,\varepsilon)} |u|.$$

Taking the limit $\varepsilon \to 0^+$, we obtain that $\partial_\mathrm{n} v(s)(\xi - u(s)) \leq |\xi| - |u(s)|$. Since this inequality is satisfied for all $\xi \in \mathbb{R}$, we deduce that $\partial_\mathrm{n} v(s) \in \partial\left|\cdot\right|(u(s))$. The proof of the second inclusion is now complete. □

Proposition 3.14. *The second-order difference quotient functions associated to the Tresca-type functional Φ satisfy*

$$\Delta_\tau^2 \Phi(u|v)(w) = \int_\Gamma \Delta_\tau^2 \left|\cdot\right| (u(s)|\partial_\mathrm{n} v(s))(w(s)) \, \mathrm{d}s,$$

for all $\tau > 0$, $u \in \mathrm{H}^1(\Omega)$, $v \in \partial\Phi(u)$ and $w \in \mathrm{H}^1(\Omega)$.

Proof. Let $\tau > 0$, $u \in \mathrm{H}^1(\Omega)$, $v \in \partial\Phi(u)$ and $w \in \mathrm{H}^1(\Omega)$. Since $v \in \partial\Phi(u)$, we know from Lemma 3.13 and the Green formula that

$$\langle v, w \rangle_{\mathrm{H}^1(\Omega)} = \int_\Gamma \partial_\mathrm{n} v \, w.$$

Thus it holds that

$$\begin{aligned} \Delta_\tau^2 \Phi(u|v)(w) = & \frac{\Phi(u + \tau w) - \Phi(u) - \tau\langle v, w\rangle_{\mathrm{H}^1(\Omega)}}{\tau^2} \\ & = \int_\Gamma \frac{|u(s) + \tau w(s)| - |u(s)| - \tau\partial_\mathrm{n} v(s) w(s)}{\tau^2} \, \mathrm{d}s. \end{aligned}$$

Since $\partial_\mathrm{n} v(s) \in \partial\left|\cdot\right|(u(s))$ for almost every $s \in \Gamma$, we exactly get the expected formula. □

Remark 3.15. If Φ is twice epi-differentiable at $u \in \mathrm{H}^1(\Omega)$ for some $v \in \partial\Phi(u)$, one can naturally expect from Proposition 3.14 that its second-order epi-derivative satisfies

$$\mathrm{d}_e^2\Phi(u|v)(w) = \int_\Gamma \mathrm{d}_e^2 \left|\cdot\right| (u(s)|\partial_\mathrm{n} v(s))(w(s)) \, \mathrm{d}s,$$

for all $w \in \mathrm{H}^1(\Omega)$. The above formula, which corresponds from Proposition 3.14 to the inversion of the ME-lim symbol and the $\int_\Gamma$ symbol, remains an open and challenging question that we postpone to a future research project. We refer to Remark 3.18 and Appendix B for the proof of the above formula in some particular cases in which u is a solution to the Tresca-type problem and $v \in \partial\Phi(u)$ is equal to the difference between a solution to a Neumann problem and u. Along the same lines let us mention the work [26] in which the author studied the inversion of the ME-lim symbol and the $\int_\Omega$ symbol over the $\mathrm{L}^2(\Omega)$-space. Nevertheless this previous work cannot be applied to our context since we consider here the $\mathrm{H}^1(\Omega)$-space and the $\int_\Gamma$ symbol over the boundary Γ (instead of the $\int_\Omega$ symbol over the set Ω in [26]).

3.4 The derivative of a parameterized Tresca-type problem

The parameterized Tresca-type problem considered in this paper is given by

$$\begin{cases} -\Delta u_t + u_t = f_t & \text{in } \Omega, \\ |\partial_\mathrm{n} u_t| \leq 1 \text{ and } u_t \partial_\mathrm{n} u_t = -|u_t| & \text{on } \Gamma, \end{cases} \tag{TP$_t$}$$

where $f_t \in \mathrm{L}^2(\Omega)$ for all $t \geq 0$. From Proposition 3.12, Problem (TP$_t$) has a unique solution $u_t \in \mathrm{H}^1(\Omega)$ given by

$$u_t = \mathrm{prox}_\Phi(F_t),$$

for all $t \geq 0$, where $F_t \in \mathrm{H}^1(\Omega)$ is the unique solution to the parameterized Neumann problem

$$\begin{cases} -\Delta F_t + F_t = f_t & \text{in } \Omega, \\ \partial_\mathrm{n} F_t = 0 & \text{on } \Gamma. \end{cases} \tag{NP$_t$}$$

In particular note that $F_0 - u_0 \in \partial\Phi(u_0)$.

Theorem 3.16. *Let us consider that the following assumptions are both satisfied:*

(A1) *the map $t \geq 0 \mapsto f_t \in \mathrm{L}^2(\Omega)$ is differentiable at $t = 0$, with derivative denoted by $f_0' \in \mathrm{L}^2(\Omega)$;*

(A2) *Φ is twice epi-differentiable at u_0 for $F_0 - u_0$ with*

$$\mathrm{d}_e^2\Phi(u_0|F_0 - u_0)(w) = \int_\Gamma \mathrm{d}_e^2 \left|\cdot\right| (u_0(s)|\partial_\mathrm{n}(F_0 - u_0)(s))(w(s)) \, \mathrm{d}s,$$

for all $w \in \mathrm{H}^1(\Omega)$.

Then the map $t \geq 0 \longmapsto u_t \in \mathrm{H}^1(\Omega)$ is differentiable at $t = 0$ and its derivative denoted by $u_0' \in \mathrm{H}^1(\Omega)$ is the unique weak solution to the Signorini-type problem given by

$$\begin{cases} -\Delta u_0' + u_0' = f_0' & \text{in } \Omega, \\ \partial_\mathrm{n} u_0' = 0 & \text{on } \Gamma_\mathrm{N}^{u_0}, \\ u_0' = 0 & \text{on } \Gamma_\mathrm{D}^{u_0}, \\ u_0' \leq 0, \ \partial_\mathrm{n} u_0' \leq 0 \textit{ and } u_0' \partial_\mathrm{n} u_0' = 0 & \text{on } \Gamma_{\mathrm{S-}}^{u_0}, \\ u_0' \geq 0, \ \partial_\mathrm{n} u_0' \geq 0 \textit{ and } u_0' \partial_\mathrm{n} u_0' = 0 & \text{on } \Gamma_{\mathrm{S+}}^{u_0}, \end{cases} \tag{SP$_0'$}$$

where

$$\Gamma_{\mathrm{N}}^{u_0} := \{s \in \Gamma \mid u_0(s) \neq 0\}, \quad \Gamma_{\mathrm{D}}^{u_0} := \{s \in \Gamma \mid u_0(s) = 0 \text{ and } \partial_{\mathrm{n}} u_0(s) \in (-1,1)\},$$

$$\Gamma_{\mathrm{S}-}^{u_0} := \{s \in \Gamma \mid u_0(s) = 0 \text{ and } \partial_{\mathrm{n}} u_0(s) = 1\}, \quad \Gamma_{\mathrm{S}+}^{u_0} := \{s \in \Gamma \mid u_0(s) = 0 \text{ and } \partial_{\mathrm{n}} u_0(s) = -1\}.$$

If moreover $\partial_{\mathrm{n}} u_0' \in \mathrm{L}^2(\Gamma)$ *and the decomposition* $\Gamma = \Gamma_{\mathrm{N}}^{u_0} \cup \Gamma_{\mathrm{D}}^{u_0} \cup \Gamma_{\mathrm{S}-}^{u_0} \cup \Gamma_{\mathrm{S}+}^{u_0}$ *is consistent, then* u_0' *is a (strong) solution to Problem* (SP_0').

Proof. From the linearity of Problem (NP_t) and from the inequality $\|F_t\|_{\mathrm{H}^1(\Omega)} \leq \|f_t\|_{\mathrm{L}^2(\Omega)}$ for all $t \geq 0$, one can easily prove that the map $t \geq 0 \longmapsto F_t \in \mathrm{H}^1(\Omega)$ is differentiable at $t = 0$ and the derivative denoted by $F_0' \in \mathrm{H}^1(\Omega)$ is the unique solution to the Neumann problem

$$\begin{cases} -\Delta F_0' + F_0' = f_0' & \text{in } \Omega, \\ \partial_{\mathrm{n}} F_0' = 0 & \text{on } \Gamma. \end{cases} \tag{NP_0'}$$

On the other hand it follows from the second hypothesis that

$$\mathrm{d}_e^2 \Phi(u_0 | F_0 - u_0)(w) = \int_\Gamma \mathrm{I}_{\mathrm{K}_{u_0(s),\partial_{\mathrm{n}}(F_0-u_0)(s)}}(w(s))\, \mathrm{d}s,$$

for all $w \in \mathrm{H}^1(\Omega)$, where the notation $\mathrm{K}_{u_0(s),\partial_{\mathrm{n}}(F_0-u_0)(s)}$ has been introduced in Example 2.6. We deduce that

$$\mathrm{d}_e^2 \Phi(u_0 | F_0 - u_0)(w) = \mathrm{I}_{\mathcal{K}_{u_0,\partial_{\mathrm{n}}(F_0-u_0)}}(w),$$

for all $w \in \mathrm{H}^1(\Omega)$, where

$$\mathcal{K}_{u_0,\partial_{\mathrm{n}}(F_0-u_0)} := \{\varphi \in \mathrm{H}^1(\Omega) \mid \varphi(s) \in \mathrm{K}_{u_0(s),\partial_{\mathrm{n}}(F_0-u_0)(s)} \text{ for almost every } s \in \Gamma\},$$

is a nonempty closed convex subset of $\mathrm{H}^1(\Omega)$. Since $\partial_{\mathrm{n}} F_0 = 0$ on Γ, note that

$$\mathcal{K}_{u_0,\partial_{\mathrm{n}}(F_0-u_0)} = \{\varphi \in \mathrm{H}^1(\Omega) \mid \varphi \leq 0 \text{ on } \Gamma_{\mathrm{S}-}^{u_0},\ \varphi = 0 \text{ on } \Gamma_{\mathrm{D}}^{u_0} \text{ and } \varphi \geq 0 \text{ on } \Gamma_{\mathrm{S}+}^{u_0}\}.$$

Finally, from Proposition 2.8, we get that the map $t \geq 0 \longmapsto u_t \in \mathrm{H}^1(\Omega)$ is differentiable at $t = 0$ and the derivative denoted by $u_0' \in \mathrm{H}^1(\Omega)$ is given by

$$u_0' = \mathrm{prox}_{\mathrm{d}_e^2 \Phi(u_0 | F_0 - u_0)}(F_0') = \mathrm{proj}_{\mathcal{K}_{u_0,\partial_{\mathrm{n}}(F_0-u_0)}}(F_0').$$

The proof is concluded by Proposition 3.6. The last part of Theorem 3.16 is due to (ii) of Proposition 3.8. □

Remark 3.17. Let us discuss some applications of our main result (Theorem 3.16). The first one is the natural approximation of the solution u_t to the perturbed Tresca-type problem (TP_t) using the solution u_0 and its derivative u_0', that is, $u_t \approx u_0 + t u_0'$ for small $t > 0$. This is the topic of the numerical simulations exposed in Section 4 below. On the other hand, Theorem 3.16 is applicable in shape optimization theory in the context of contact mechanics. Indeed the sensitivity analysis with respect to the shape of a cost functional requires to compute and characterize the shape derivative of the state. Hence, in view of shape optimization problems involving Tresca-type problems, our main result would allow to characterize the shape derivative of the state and thus to obtain a useful expression of the shape gradient of the cost functional in view of numerical approximations. This nontrivial application to shape optimization problems has been our primary motivation and will be the subject of a forthcoming work by the authors.

Remark 3.18. This remark is dedicated to Assumption (A2) which is not trivial as evoked in Remark 3.15. In this discussion we will consider the notation $\mathcal{K}_{u_0,\partial_{\mathrm{n}}(F_0-u_0)}$ introduced in the proof of Theorem 3.16. From Proposition 2.3, Assumption (A2) requires that

(i) for all $w \in \mathrm{H}^1(\Omega)$ and all $(w_\tau)_{\tau>0} \subset \mathrm{H}^1(\Omega)$ such that $(w_\tau)_{\tau>0} \rightharpoonup w$ in $\mathrm{H}^1(\Omega)$, it holds that

$$\liminf \Delta_\tau^2 \Phi(u_0|F_0-u_0)(w_\tau) \geq \mathrm{I}_{\mathcal{K}_{u_0,\partial_{\mathrm{n}}(F_0-u_0)}}(w),$$

which can be rewritten as

$$\liminf \int_\Gamma \Delta_\tau^2 \left|\cdot\right| (u_0(s)|\partial_{\mathrm{n}}(F_0-u_0)(s))(w_\tau(s))\,\mathrm{d}s \\ \geq \int_\Gamma \mathrm{d}_e^2 \left|\cdot\right| (u_0(s)|\partial_{\mathrm{n}}(F_0-u_0)(s))(w(s))\,\mathrm{d}s,$$

from Proposition 3.14;

(ii) and for all $w \in \mathrm{H}^1(\Omega)$, there exists $(w_\tau)_{\tau>0} \subset \mathrm{H}^1(\Omega)$ such that $(w_\tau)_{\tau>0} \to w$ in $\mathrm{H}^1(\Omega)$ and

$$\limsup \Delta_\tau^2 \Phi(u_0|F_0-u_0)(w_\tau) \leq \mathrm{I}_{\mathcal{K}_{u_0,\partial_{\mathrm{n}}(F_0-u_0)}}(w).$$

From the continuous compact embedding $\mathrm{H}^1(\Omega) \hookrightarrow \mathrm{L}^2(\Gamma)$, the classical Fatou lemma (see, e.g., [7, Lemma 4.1 p.90]) and the twice epi-differentiability of the absolute value map $|\cdot|$ (see Example 2.6), the point (i) is obviously satisfied. The point (ii) is trivial for $w \in \mathrm{H}^1(\Omega)$ such that $w \notin \mathcal{K}_{u_0,\partial_{\mathrm{n}}(F_0-u_0)}$. Finally, in order to check the validity of Assumption (A2), one has (only) to prove that

(ii') for all $w \in \mathcal{K}_{u_0,\partial_{\mathrm{n}}(F_0-u_0)}$, there exists $(w_\tau)_{\tau>0} \subset \mathrm{H}^1(\Omega)$ such that $(w_\tau)_{\tau>0} \to w$ in $\mathrm{H}^1(\Omega)$ and

$$\limsup \Delta_\tau^2 \Phi(u_0|F_0-u_0)(w_\tau) \leq 0.$$

In the case where $\Gamma_{\mathrm{N}}^{u_0}$ has a null measure, the point (ii') can be easily derived by taking the constant sequence $w_\tau := w$ for all $\tau > 0$, and thus Assumption (A2) is satisfied. However we are not able to write a general proof of the point (ii') in the case where $\Gamma_{\mathrm{N}}^{u_0}$ has a positive measure. Nevertheless, in Appendix B, we provide some examples of sufficient conditions on u_0 and Γ in order to guarantee that the point (ii') is satisfied, even if $\Gamma_{\mathrm{N}}^{u_0}$ has a positive measure.

4 Illustration with some numerical simulations

In what follows we preserve the notations introduced in Section 3. Our aim in this section is to illustrate our main result (Theorem 3.16) with some numerical simulations. Let us mention that the numerical simulations are performed using Freefem++ software (see [18]) and that the iterative switching algorithms are presented in Appendix C. In full agreement with the main result of the present work, these simulations underline that, for a given small $t > 0$, the unique solution u_t to the parameterized Tresca-type problem (TP_t) can be approximated by $u_0 + tu_0'$, where u_0' is the unique weak solution to the Signorini-type problem (SP_0').

Two-dimensional simulations. Let $d = 2$ and Ω be the unit open disk of $\mathbb{R}^2$. Then, for all $t \geq 0$, we consider the function $f_t \in \mathrm{L}^2(\Omega)$ defined by $f_t(x,y) := e^t f(x,y)$ where

$$f(x,y) := \frac{1}{2}(x^2+y^2-5)h(x) - 2xh'(x) - \frac{1}{2}(x^2+y^2-1)h''(x),$$

for almost all $(x,y) \in \Omega$, where

$$h(x) := \begin{cases} -1 & \text{if} \quad -1 < x \leq -\frac{1}{2}, \\ \sin(\pi x) & \text{if} \quad -\frac{1}{2} \leq x \leq \frac{1}{2}, \\ 1 & \text{if} \quad \frac{1}{2} \leq x < 1, \end{cases}$$

for all $x \in (-1,1)$. Our choice of such a function $f = f_0$ is justified by the fact that we are able, in this case, to express analytically the exact solution u_0 to Problem (TP_0), which is given by

$$u_0(x,y) := \frac{1}{2}(x^2+y^2-1)h(x),$$

for all $(x,y) \in \Omega$. On the other hand the choice of the expression of the function h is justified by the fact that it provides an example in which the decomposition

$$\Gamma = \Gamma_{\mathrm{N}}^{u_0} \cup \Gamma_{\mathrm{D}}^{u_0} \cup \Gamma_{\mathrm{S}-}^{u_0} \cup \Gamma_{\mathrm{S}+}^{u_0},$$

is nontrivial in the sense that $\Gamma_{\mathrm{S}-}^{u_0} \cup \Gamma_{\mathrm{S}+}^{u_0}$ has a positive measure, which guarantees in the Signorini-type problem (SP_0') the presence of parts of the boundary with Signorini's conditions. Indeed, one can easily deduce from the expression of u_0 that

$$\Gamma_{\mathrm{S}+}^{u_0} = \left\{ (x,y) \in \Gamma \mid x \leq -\frac{1}{2} \right\}, \quad \Gamma_{\mathrm{D}}^{u_0} = \left\{ (x,y) \in \Gamma \mid -\frac{1}{2} < x < \frac{1}{2} \right\},$$
$$\text{and} \quad \Gamma_{\mathrm{S}-}^{u_0} = \left\{ (x,y) \in \Gamma \mid x \geq \frac{1}{2} \right\}.$$

We refer to Figure 1. In order to illustrate Theorem 3.16, we first compute numerically the solutions u_0 and u_0'. Then, for several small values $t > 0$, we compute numerically the solution u_t and compare it with $u_0 + tu_0'$ (using the H^1-norm). We concatenate our results in Table 1, and Figure 2 gives the representation of the H^1-comparison with respect to t in logarithmic scale. Figure 3 concludes this paragraph with the illustration of the case $t = 0.01$.

t	0.40	0.20	0.15	0.1	0.075	0.05	0.025	0.01	0.0075	0.005	0.0025
$\Vert u_t - u_0 - tu_0' \Vert_{\mathrm{H}^1(\Omega)}$	0.6528	0.2360	0.1580	0.0909	0.0616	0.0360	0.0138	0.0040	0.0029	0.0021	0.0016

Table 1: H^1-norm of the difference between u_t and its first-order approximation $u_0 + tu_0'$.

Three-dimensional simulations. Let $d = 3$ and Ω be the cube $(0,1)^3$. We use here the parameterized function $f_t \in \mathrm{L}^2(\Omega)$ (chosen haphazardly) defined by

$$f_t(x,y,z) := \sin(t)xe^y e^{2z} + \sqrt{x}\cos(xy^2)e^z + 25(z-1),$$

for all $(x,y,z) \in \Omega$ and all $t \geq 0$. Figure 4 illustrates the solution u_t for $t = 0.1$ and its first-order approximation $u_0 + tu_0'$. Here we obtain $\Vert u_t - u_0 - tu_0' \Vert_{\mathrm{H}^1(\Omega)} = 0.000575736$.

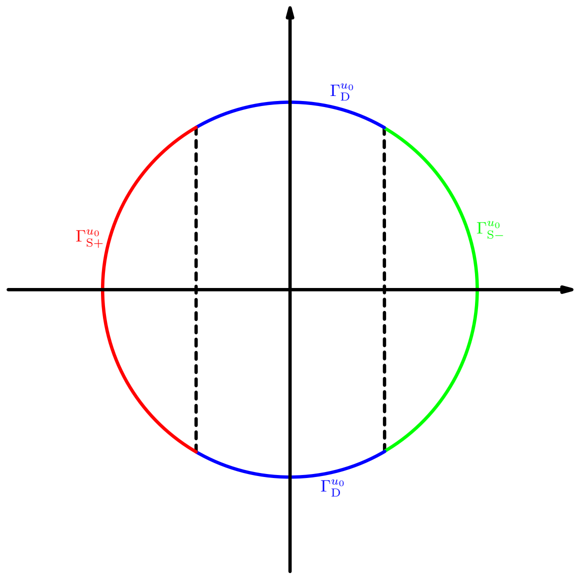

Figure 1: Boundary decomposition for the two-dimensional example of Section 4.

5 Concluding remarks

In this paper we investigated the sensitivity analysis of a scalar Tresca-type problem. The sensitivity analysis in our context has to be understood in the sense of the differentiable property of the solution with respect to right-hand source term perturbations. Using second-order variational analysis tools, and particularly the sophisticated concept of twice epi-differentiability of a proper closed convex function, we show that the derivative of the solution to the parameterized Tresca problem satisfies Signorini's conditions. At first glance, the Tresca-type problem and the Signorini-type problem seem to have no connection between them. Thanks to Theorem 3.16, we can roughly say, in our context, that Signorini's solutions can be considered as first-order approximations of Tresca's solutions in a certain sense. This fact was highlighted by the numerical simulations in Section 4. The combination of tools from both functional and convex analyses was fruitful in the context of this paper and permit us to obtain original results. In fact, the concept of twice epi-differentiability is usually used in the optimization community in finite-dimensional spaces. Applying it for other problems (in particular in infinite-dimensional settings) opens the way to disseminate this concept to other communities. Many questions need further investigations such as the case where all the data are perturbed (including the friction threshold for instance), or giving sufficient conditions for the twice epi-differentiability of the Tresca-type functional Φ (see Remark 3.18 and Appendix B). It is worth noticing that we focused here on the scalar case but we are confident that our methodology can be extended in the same manner to the linear elasticity case. Also it is well-known that the Tresca friction law is an approximation of the more realistic Coulomb one. This may open possibilities for further extensions to quasi-variational inequalities and to time-dependent processes in contact mechanics. This is out of the scope of the current paper and will be the subject of forthcoming research projects.

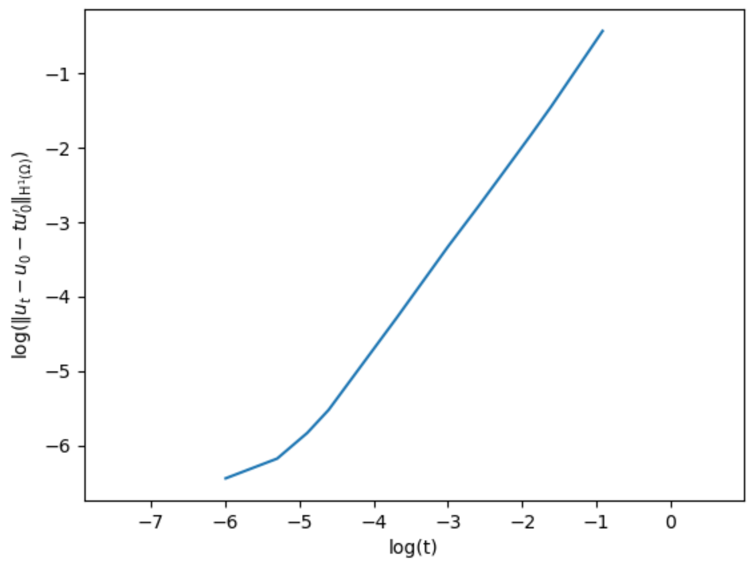

Figure 2: The H^1-comparison $\|u_t - u_0 - tu_0'\|_{\mathrm{H}^1(\Omega)}$ with respect to t in logarithmic scale.

A A result on the extension of a linear operator

Let V (resp. H) be a real Hilbert space. We denote by $\langle\cdot,\cdot\rangle_{\mathrm{V}}$ (resp. $\langle\cdot,\cdot\rangle_{\mathrm{H}}$) the corresponding scalar product and by $\|\cdot\|_{\mathrm{V}}$ (resp. $\|\cdot\|_{\mathrm{H}}$) the associated norm. We assume that the continuous and dense embedding $\mathrm{V} \hookrightarrow \mathrm{H}$ holds. We introduce V^* as the completion of H for the norm $\|\cdot\|_*$ defined on H by

$$\|h\|_* := \sup_{\substack{v\in\mathrm{V}\\ \|v\|_{\mathrm{V}}\leq 1}} |\langle h,v\rangle_{\mathrm{H}}|\,,$$

for all $h \in \mathrm{H}$. Then we define the operator $J : \mathrm{V}^* \to \mathrm{V}'$, where V' stands for the dual space of V, by

$$\forall h\in\mathrm{V}^*,\quad \forall v\in\mathrm{V},\quad \langle Jh,v\rangle_{\mathrm{V}'\times\mathrm{V}} := \lim_{n\to\infty}\langle h_n,v\rangle_{\mathrm{H}},\ \text{ where } (h_n)_n\subset\mathrm{H} \text{ such that } h_n\to h \text{ in } \mathrm{V}^*.$$

Recall that J is an isomorphism (see, e.g., [36, Proposition 2.9.2 p.56]) and thus we can identify V^* and V' (without distinction between h and Jh for all $h \in \mathrm{V}^*$). We finally have

$$\mathrm{V} \underset{\text{dense}}{\hookrightarrow} \mathrm{H} \underset{\text{dense}}{\hookrightarrow} \mathrm{V}',$$

and, in that context, H is usually called the *pivot space*.

Proposition A.1. *Consider the above framework and let $w \in \mathrm{V}'$. If*

$$\exists c\geq 0,\quad \forall v\in\mathrm{V},\quad \langle w,v\rangle_{\mathrm{V}'\times\mathrm{V}} \leq c\,\|v\|_{\mathrm{H}}\,,$$

then w can be identified to an element $h \in \mathrm{H}$ with $\langle w,v\rangle_{\mathrm{V}'\times\mathrm{V}} = \langle h,v\rangle_{\mathrm{H}}$ for all $v \in \mathrm{V}$.

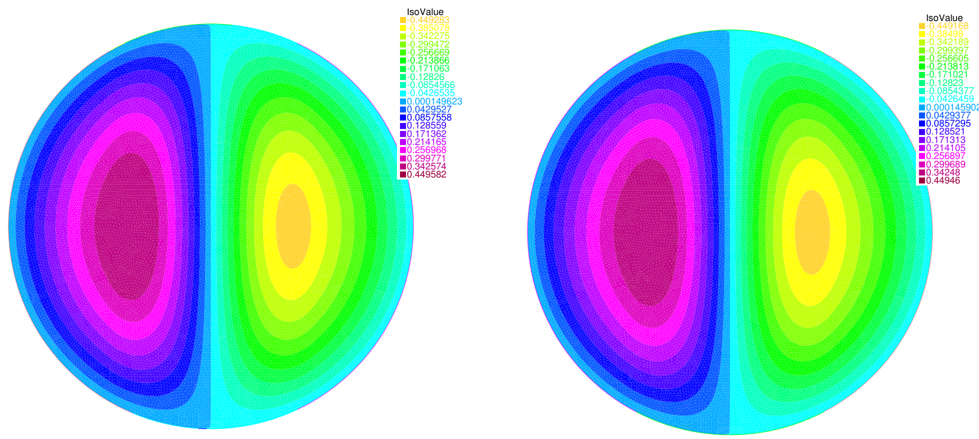

Figure 3: Case $t = 0.01$ ($d = 2$): solution u_t (left) and its first-order approximation $u_0 + tu_0'$ (right).

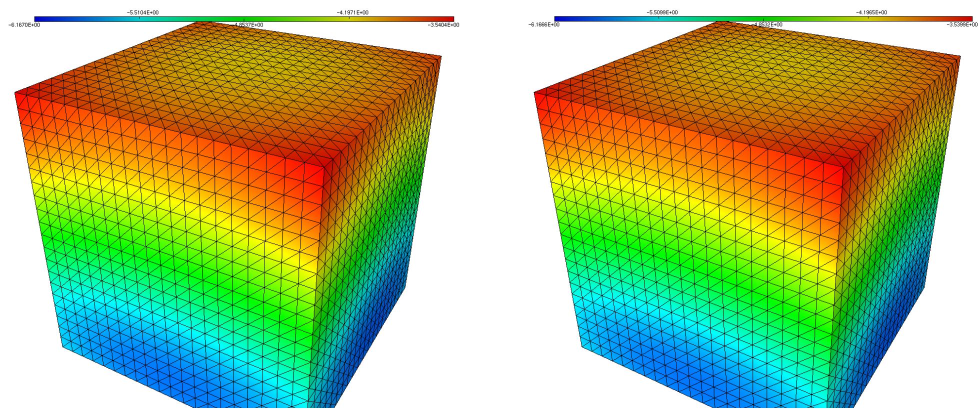

Figure 4: Case $t = 0.1$ ($d = 3$): solution u_t (left) and its first-order approximation $u_0 + tu_0'$ (right).

Proof. From the hypothesis and [7, Corollary 1.2], there exists $h \in \mathrm{H}$ such that $\langle w, v\rangle_{\mathrm{V}'\times\mathrm{V}} = \langle h, v\rangle_{\mathrm{H}}$ for all $v \in \mathrm{V}$. Using the above definition of the functional J (and taking the constant sequence $(h_n)_n$ equal to h), we also have $\langle Jh, v\rangle_{\mathrm{V}'\times\mathrm{V}} = \langle h, v\rangle_{\mathrm{H}}$ for all $v \in \mathrm{V}$, and thus $\langle w, v\rangle_{\mathrm{V}'\times\mathrm{V}} = \langle Jh, v\rangle_{\mathrm{V}'\times\mathrm{V}}$ for all $v \in \mathrm{V}$. Hence $w = Jh$ in V' and thus w can be identified to h with $\langle w, v\rangle_{\mathrm{V}'\times\mathrm{V}} = \langle h, v\rangle_{\mathrm{H}}$ for all $v \in \mathrm{V}$. $\square$

B Some sufficient conditions for the point (ii’) of Remark 3.18

In this appendix we keep the notation introduced in Section 3 and we introduce the sets

$$\Gamma_{\mathrm{N}-}^{u_0} := \{s \in \Gamma \mid u_0(s) < 0\} \quad \text{and} \quad \Gamma_{\mathrm{N}+}^{u_0} := \{s \in \Gamma \mid u_0(s) > 0\}.$$

In particular it holds that $\Gamma_{\mathrm{N}}^{u_0} = \Gamma_{\mathrm{N}-}^{u_0} \cup \Gamma_{\mathrm{N}+}^{u_0}$ and thus the (pairwise disjoint) decomposition

$$\Gamma = \Gamma_{\mathrm{N}-}^{u_0} \cup \Gamma_{\mathrm{N}+}^{u_0} \cup \Gamma_{\mathrm{D}}^{u_0} \cup \Gamma_{\mathrm{S}-}^{u_0} \cup \Gamma_{\mathrm{S}+}^{u_0},$$

holds true. Our aim in this section is to provide some examples of sufficient condition on u_0 and Γ which ensures that the point (ii') of Remark 3.18 is satisfied, even if $\Gamma_{\mathrm{N}}^{u_0}$ has a positive measure.

A first (and simple) example of sufficient condition. In this paragraph we assume that $\Gamma = \Gamma_{\mathrm{N}+}^{u_0}$ and that u_0 is continuous over the compact set Γ. In particular it holds that $u_0(s) \geq c$ for all $s \in \Gamma$ for some $c > 0$. Let $w \in \mathcal{K}_{u_0,\partial_{\mathrm{n}}(F_0-u_0)} = \mathrm{H}^1(\Omega)$ and let us prove that there exists $(w_\tau)_{\tau>0} \subset \mathrm{H}^1(\Omega)$ such that $(w_\tau)_{\tau>0} \to w$ in $\mathrm{H}^1(\Omega)$ and

$$\limsup \Delta_\tau^2 \Phi(u_0|F_0 - u_0)(w_\tau) \leq 0.$$

To this aim we consider the *truncature* $w_\tau \in \mathrm{H}^1(\Omega)$ of w defined by

$$w_\tau(x) := \begin{cases} \frac{1}{\sqrt{\tau}} & \text{if} \quad w(x) \geq \frac{1}{\sqrt{\tau}}, \\ w(x) & \text{if} \quad -\frac{1}{\sqrt{\tau}} \leq w(x) \leq \frac{1}{\sqrt{\tau}}, \\ -\frac{1}{\sqrt{\tau}} & \text{if} \quad w(x) \leq -\frac{1}{\sqrt{\tau}}, \end{cases}$$

for almost all $x \in \Omega$ and all $\tau > 0$. It is clear that $w_\tau \to w$ in $\mathrm{H}^1(\Omega)$ when $\tau \to 0^+$. Moreover, from the inequality $u_0 \geq c$ and the equalities $\partial_{\mathrm{n}} u_0 = -1$ and $\partial_{\mathrm{n}} F_0 = 0$ over Γ, we get from Proposition 3.14 that

$$\Delta_\tau^2 \Phi(u_0|F_0 - u_0)(w_\tau) = \int_\Gamma \frac{|u_0(s) + \tau w_\tau(s)| - |u_0(s)| - \tau \partial_{\mathrm{n}}(F_0 - u_0)(s) w_\tau(s)}{\tau^2} \, \mathrm{d}s = 0,$$

for all $\tau > 0$ sufficiently small. The proof is thereby completed. Obviously this strategy can be adapted to the case where $\Gamma = \Gamma_{\mathrm{N}-}^{u_0}$ and u_0 is continuous over Γ.

A second example of sufficient condition in the two-dimensional case $d = 2$. In this paragraph we consider the two-dimensional case $d = 2$. In that context Γ is a single closed curve. We assume that Γ is a smooth curve and we denote by $\gamma : \mathbb{R} \to \Gamma$ an absolutely continuous T-periodic parameterization of Γ, where $T > 0$ can be chosen as desired, and by $\dot{\gamma} : \mathbb{R} \to \mathbb{R}^2$ its essentially bounded derivative. In particular it holds that $\Gamma = \{\gamma(r) \mid r \in [0, T]\}$. Let us assume that the (pairwise disjoint) decomposition $\Gamma = \Gamma_{\mathrm{N}-}^{u_0} \cup \Gamma_{\mathrm{N}+}^{u_0} \cup \Gamma_{\mathrm{D}}^{u_0} \cup \Gamma_{\mathrm{S}-}^{u_0} \cup \Gamma_{\mathrm{S}+}^{u_0}$ satisfies

$$\Gamma_{\mathrm{N}-}^{u_0} := \bigcup_{j=1}^{k_1} \{\gamma(r) \mid r_j^{k_1} < r < r_{j+1}^{k_1}\}, \quad \Gamma_{\mathrm{N}+}^{u_0} := \bigcup_{j=1}^{k_2} \{\gamma(r) \mid r_j^{k_2} < r < r_{j+1}^{k_2}\},$$

$$\Gamma_{\mathrm{D}}^{u_0} := \bigcup_{j=1}^{k_3} \{\gamma(r) \mid r_j^{k_3} < r < r_{j+1}^{k_3}\},$$

$$\Gamma_{\mathrm{S}-}^{u_0} := \bigcup_{j=1}^{k_4} \{\gamma(r) \mid r_j^{k_4} \leq r \leq r_{j+1}^{k_4}\}, \quad \Gamma_{\mathrm{S}+}^{u_0} := \bigcup_{j=1}^{k_5} \{\gamma(r) \mid r_j^{k_5} \leq r \leq r_{j+1}^{k_5}\},$$

where $k_i \in \mathbb{N}$ for all $i \in \{1,\ldots,5\}$ (with $k_1 + k_2 \geq 1$) and where $r_j^{k_i} < r_{j+1}^{k_i}$ for all $j \in \{1,\ldots,k_i\}$ and all $i \in \{1,\ldots,5\}$, with

$$\left(\bigcup_{i=1}^{3} \bigcup_{j=1}^{k_i} (r_j^{k_i}, r_{j+1}^{k_i}) \right) \bigcup \left(\bigcup_{i=4}^{5} \bigcup_{j=1}^{k_i} [r_j^{k_i}, r_{j+1}^{k_i}] \right) = [0,T].$$

Finally we also make the quite natural assumption (which is true for example when u_0 and $\partial_{\mathrm{n}} u_0$ are continuous over Γ) that the decomposition is such that a part of $\Gamma_{\mathrm{N-}}^{u_0}$ is always side to side with a part of $\Gamma_{\mathrm{D}}^{u_0}$ or with a part of $\Gamma_{\mathrm{S-}}^{u_0}$, and similarly that $\Gamma_{\mathrm{N+}}^{u_0}$ is always side to side with a part of $\Gamma_{\mathrm{D}}^{u_0}$ or with a part of $\Gamma_{\mathrm{S+}}^{u_0}$. We refer to Figure 5 below for an illustration of an admissible decomposition.

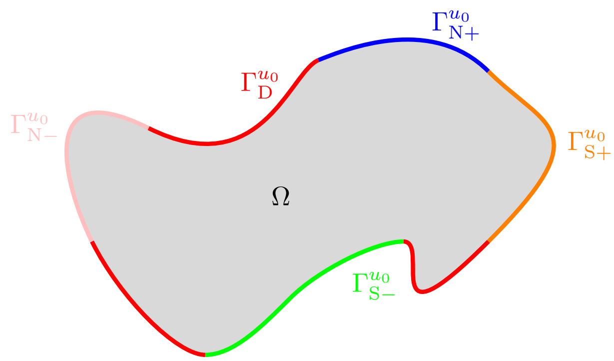

Figure 5: Illustration on an admissible decomposition of Γ.

As in the previous paragraph we will assume that u_0 is continuous over Γ. In that context we assert that the point (ii') of Remark 3.15 is satisfied. For the ease of presentation and notation, we only give the proof in the case where $k_2 = k_5 = 1$ and $k_1 = k_3 = k_4 = 0$ (nonetheless one can easily understand that the proof below can be extended to the general case). We choose $T = 6$ and, in that context, we can write $\Gamma = \Gamma_{\mathrm{N+}}^{u_0} \cup \Gamma_{\mathrm{S+}}^{u_0}$ with

$$\Gamma_{\mathrm{S+}}^{u_0} := \{\gamma(r) \mid 0 \leq r \leq 3\} \quad \text{and} \quad \Gamma_{\mathrm{N+}}^{u_0} := \{\gamma(r) \mid 3 < r < 6\}. \tag{2}$$

In particular, from continuity of u_0 over Γ, note that $u_0(\gamma(r)) \geq c$ for all $r \in [4,5]$ for some $c > 0$. We refer to Figure 6 below for an illustration of the notation concerning the parameterization.

Let $w \in \mathcal{K}_{u_0,\partial_{\mathrm{n}}(F_0 - u_0)}$ and let us prove that there exists $(w_\tau)_{\tau>0} \subset \mathrm{H}^1(\Omega)$ such that $(w_\tau)_{\tau>0} \to w$ in $\mathrm{H}^1(\Omega)$ and

$$\limsup \Delta_\tau^2 \Phi(u_0|F_0 - u_0)(w_\tau) \leq 0.$$

To this aim we first consider the *truncature* $y_\tau \in \mathrm{H}^1(\Omega)$ of w defined by

$$y_\tau(x) := \begin{cases} \frac{1}{\sqrt{\tau}} & \text{if} \quad w(x) \geq \frac{1}{\sqrt{\tau}}, \\ w(x) & \text{if} \quad -\frac{1}{\sqrt{\tau}} \leq w(x) \leq \frac{1}{\sqrt{\tau}}, \\ -\frac{1}{\sqrt{\tau}} & \text{if} \quad w(x) \leq -\frac{1}{\sqrt{\tau}}, \end{cases}$$

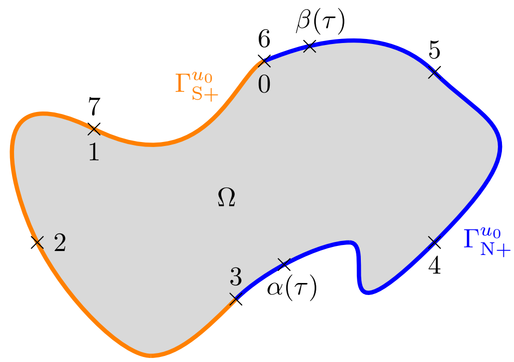

Figure 6: Illustration of the parameterization (2) of Γ.

for all $\tau > 0$ and almost all $x \in \Omega$. It is clear that $y_\tau \to w$ in $\mathrm{H}^1(\Omega)$ when $\tau \to 0^+$. For all $\tau > 0$ sufficiently small (such that $\sqrt{\tau} \leq c$), we define

$$\alpha(\tau) := \inf\{\alpha \in [3,4] \mid \forall r \in [\alpha, 4],\ u_0(\gamma(r)) \geq \sqrt{\tau}\},$$

and

$$\beta(\tau) := \sup\{\beta \in [5,6] \mid \forall r \in [5, \beta],\ u_0(\gamma(r)) \geq \sqrt{\tau}\}.$$

From continuity of u_0 and since $u_0(\gamma(3)) = u_0(\gamma(6)) = 0$, we deduce that $\alpha(\tau) \to 3$ and $\beta(\tau) \to 6$ when $\tau \to 0^+$. Then, for all $\tau > 0$ sufficiently small, we consider the *dilatation* $z_\tau \in \mathrm{H}^{1/2}(\Gamma)$ of $y_{\tau|\Gamma}$ defined by

$$\forall r \in [0,6], \quad z_\tau(\gamma(r)) := \begin{cases} y_\tau(\gamma(r)) & \text{if} \quad r \in [1,2], \\ y_\tau\left(\gamma\left(\dfrac{r + 2(\alpha(\tau)-3)}{\alpha(\tau)-2}\right)\right) & \text{if} \quad r \in [2, \alpha(\tau)], \\ y_\tau\left(\gamma\left(\dfrac{r + 4(3-\alpha(\tau))}{4-\alpha(\tau)}\right)\right) & \text{if} \quad r \in [\alpha(\tau), 4], \\ y_\tau(\gamma(r)) & \text{if} \quad r \in [4,5], \\ y_\tau\left(\gamma\left(\dfrac{r + 5(\beta(\tau)-6)}{\beta(\tau)-5}\right)\right) & \text{if} \quad r \in [5, \beta(\tau)], \\ y_\tau\left(\gamma\left(\dfrac{r + 7(6-\beta(\tau))}{7-\beta(\tau)}\right)\right) & \text{if} \quad r \in [\beta(\tau), 7]. \end{cases}$$

Roughly speaking we have constructed z_τ such that the graph of $z_\tau \circ \gamma$ over $[2, \alpha(\tau)]$ corresponds to the graph of $y_\tau \circ \gamma$ over $[2,3]$ (and similarly on the other intervals). It holds that $z_\tau \to w_{|\Gamma}$ in $\mathrm{H}^{1/2}(\Gamma)$ when $\tau \to 0^+$. Then, for all $\tau > 0$ sufficiently small, we denote by $w_\tau \in \mathrm{H}^1(\Omega)$ a lift of $z_\tau \in \mathrm{H}^{1/2}(\Gamma)$ which satisfies $w_\tau \to w$ in $\mathrm{H}^1(\Omega)$ when $\tau \to 0^+$. Since $\partial_{\mathrm{n}} F_0 = 0$ over Γ and denoting by

$$m_\tau(s) := \Delta_\tau^2 |\cdot|(u_0(s)|\partial_{\mathrm{n}}(F_0 - u_0)(s))(w_\tau(s)) = \Delta_\tau^2 |\cdot|(u_0(s)|\partial_{\mathrm{n}}(F_0 - u_0)(s))(z_\tau(s))$$
$$= \frac{|u_0(s) + \tau z_\tau(s)| - |u_0(s)| - \tau \partial_{\mathrm{n}}(F_0 - u_0)(s) z_\tau(s)}{\tau^2}$$

$$= \frac{|u_0(s) + \tau z_\tau(s)| - |u_0(s)| + \tau \partial_{\mathrm{n}} u_0(s) z_\tau(s)}{\tau^2},$$

for all $\tau > 0$ sufficiently small and almost all $s \in \Gamma$, we get from Proposition 3.14 that

$$\Delta_\tau^2 \Phi(u_0|F_0 - u_0)(w_\tau) = \int_\Gamma m_\tau(s) \, \mathrm{d}s = \int_1^2 \|\dot{\gamma}(r)\|_{\mathbb{R}^2} m_\tau(\gamma(r)) \, \mathrm{d}r + \int_2^{\alpha(\tau)} \|\dot{\gamma}(r)\|_{\mathbb{R}^2} m_\tau(\gamma(r)) \, \mathrm{d}r$$
$$+ \int_{\alpha(\tau)}^4 \|\dot{\gamma}(r)\|_{\mathbb{R}^2} m_\tau(\gamma(r)) \, \mathrm{d}r + \int_4^5 \|\dot{\gamma}(r)\|_{\mathbb{R}^2} m_\tau(\gamma(r)) \, \mathrm{d}r$$
$$+ \int_5^{\beta(\tau)} \|\dot{\gamma}(r)\|_{\mathbb{R}^2} m_\tau(\gamma(r)) \, \mathrm{d}r + \int_{\beta(\tau)}^7 \|\dot{\gamma}(r)\|_{\mathbb{R}^2} m_\tau(\gamma(r)) \, \mathrm{d}r.$$

From the definitions of $\alpha(\tau)$, $\beta(\tau)$ and z_τ, we get that each of the six above integrands (and thus the integrals) are equal to zero. We deduce that $\Delta_\tau^2 \Phi(u_0|F_0 - u_0)(w_\tau) = 0$ for all $\tau > 0$ sufficiently small, and thus $\limsup \Delta_\tau^2 \Phi(u_0|F_0 - u_0)(w_\tau) \leq 0$. The proof is complete.

Remark B.1. The above strategy provides in the two-dimensional case $d = 2$ a quite general sufficient condition on u_0 and Γ which guarantees that the point (ii') of Remark 3.18 is satisfied, even if $\Gamma_{\mathrm{N}}^{u_0}$ has a positive measure. Although we are rather confident that this method could be extended to the three-dimensional case $d = 3$, it seems clear to us that several technical difficulties should be overcome concerning the *dilatation procedure*. For example one would need to assume that each part of the (pairwise disjoint) decomposition $\Gamma = \Gamma_{\mathrm{N}-}^{u_0} \cup \Gamma_{\mathrm{N}+}^{u_0} \cup \Gamma_{\mathrm{D}}^{u_0} \cup \Gamma_{\mathrm{S}-}^{u_0} \cup \Gamma_{\mathrm{S}+}^{u_0}$ is star-shaped and would need to introduce a corresponding and adapted (and probably not trivial) dilatation procedure on the two-dimensional manifold Γ.

Remark B.2. We conclude this section by emphasizing that the proof of the point (ii') of Remark 3.15 in a general setting (that is, without any assumption on u_0 and Γ and in any dimension $d \in \mathbb{N}^*$) remains an open challenge. Although we are not able to provide such a proof yet, we conjecture that this result is true. In that case Assumption (A2) in Theorem 3.16 would be superfluous.

C Iterative switching algorithms used in Section 4

In the literature numerous algorithms are dedicated to the numerical approximation of variational inequalities. Among others we can mention the standard penalization techniques (see, e.g., the book of N. Kikuchi and J.T. Oden [20]), mixed methods (see, e.g., the book of J. Haslinger *et al.* [17]), the hybrid methods (see, e.g., the work of F. Ben Belgacem and Y. Renard [6]), the stabilized Lagrange multiplier method (see, e.g., the work of P. Hild and Y. Renard [19]), or the Nitsche method (see, e.g., the work of F. Chouly and P. Hild [8]). In order to solve Signorini-type problems, we used in Section 4 the iterative switching algorithm introduced by J.M. Aitchison and M.W. Poole in [2], which is recalled below for the reader's convenience. Then, being inspired by this procedure, we propose hereafter an adapted iterative switching algorithm in order to solve numerically Tresca-type problems. We bring to the attention of the reader that it is not our purpose in the present work to fully analyze this algorithm and to compare it with the previously mentioned ones. Our aim here is (only) to provide a simple and easily implementable method to solve Tresca-type problems. Nevertheless, the considered algorithm is experimentally validated by the example introduced in Section 4 for which the explicit expression of the exact solution is known.

Iterative switching algorithm for the Signorini-type problem. We recall in this paragraph the algorithm proposed in [2] in order to solve the Signorini-type problem (SP) given by

$$\begin{cases} -\Delta u + u = f & \text{in } \Omega, \\ \partial_{\mathrm{n}} u = 0 & \text{on } \Gamma_{\mathrm{N}}, \\ u = 0 & \text{on } \Gamma_{\mathrm{D}}, \\ u \leq 0,\ \partial_{\mathrm{n}} u \leq 0 \text{ and } u\partial_{\mathrm{n}} u = 0 & \text{on } \Gamma_{\mathrm{S}-}, \\ u \geq 0,\ \partial_{\mathrm{n}} u \geq 0 \text{ and } u\partial_{\mathrm{n}} u = 0 & \text{on } \Gamma_{\mathrm{S}+}. \end{cases} \tag{SP}$$

Roughly speaking the algorithm is based on the fact that we need $u = 0$ or $\partial_{\mathrm{n}} u = 0$ on $\Gamma_{\mathrm{S}-} \cup \Gamma_{\mathrm{S}+}$. The main idea is to impose Neumann or Dirichlet boundary conditions on $\Gamma_{\mathrm{S}-}$ and $\Gamma_{\mathrm{S}+}$, and then to check if the Signorini conditions are satisfied. If not, we change the Dirichlet boundary into a Neumann boundary and conversely. This permits to obtain an iterative algorithm whose proof of convergence is given in [2]. More precisely we use the following strategy:

Step 1. Solve the following well-posed problem

$$\begin{cases} -\Delta \phi_0 + \phi_0 &=& f & \text{in } \Omega, \\ \partial_{\mathrm{n}} \phi_0 &=& 0 & \text{on } \Gamma_{\mathrm{N}}, \\ \phi_0 &=& 0 & \text{on } \Gamma_{\mathrm{D}}, \\ \phi_0 &=& 0 & \text{on } \Gamma_{\mathrm{S}-}, \\ \phi_0 &=& 0 & \text{on } \Gamma_{\mathrm{S}+}. \end{cases}$$

If $\partial_{\mathrm{n}} \phi_0 \leq 0$ on $\Gamma_{\mathrm{S}-}$ and $\partial_{\mathrm{n}} \phi_0 \geq 0$ on $\Gamma_{\mathrm{S}+}$, then ϕ_0 is the solution to Problem (SP). Otherwise, go to Step 2.

Step 2. If there exists $\Gamma^1_{\mathrm{S}-,\mathrm{N}} \subset \Gamma_{\mathrm{S}-}$ such that $\partial_{\mathrm{n}} \phi_0 > 0$, we define $\Gamma^1_{\mathrm{S}-,\mathrm{D}} := \Gamma_{\mathrm{S}-} \backslash \overline{\Gamma^1_{\mathrm{S}-,\mathrm{N}}}$. Similarly, if there exists $\Gamma^1_{\mathrm{S}+,\mathrm{N}} \subset \Gamma_{\mathrm{S}+}$ such that $\partial_{\mathrm{n}} \phi_0 < 0$, we define $\Gamma^1_{\mathrm{S}+,\mathrm{D}} := \Gamma_{\mathrm{S}+} \backslash \overline{\Gamma^1_{\mathrm{S}+,\mathrm{N}}}$. Then solve

$$\begin{cases} -\Delta \phi_1 + \phi_1 &=& f & \text{in } \Omega, \\ \partial_{\mathrm{n}} \phi_1 &=& 0 & \text{on } \Gamma_{\mathrm{N}}, \\ \phi_1 &=& 0 & \text{on } \Gamma_{\mathrm{D}}, \\ \phi_1 &=& 0 & \text{on } \Gamma^1_{\mathrm{S}-,\mathrm{D}} \subset \Gamma_{\mathrm{S}-}, \\ \partial_{\mathrm{n}} \phi_1 &=& 0 & \text{on } \Gamma^1_{\mathrm{S}-,\mathrm{N}} \subset \Gamma_{\mathrm{S}-}, \\ \phi_1 &=& 0 & \text{on } \Gamma^1_{\mathrm{S}+,\mathrm{D}} \subset \Gamma_{\mathrm{S}+}, \\ \partial_{\mathrm{n}} \phi_1 &=& 0 & \text{on } \Gamma^1_{\mathrm{S}+,\mathrm{N}} \subset \Gamma_{\mathrm{S}+}. \end{cases}$$

If $\partial_{\mathrm{n}} \phi_1 \leq 0$ on $\Gamma^1_{\mathrm{S}-,\mathrm{D}}$, $\phi_1 \leq 0$ on $\Gamma^1_{\mathrm{S}-,\mathrm{N}}$, $\partial_{\mathrm{n}} \phi_1 \geq 0$ on $\Gamma^1_{\mathrm{S}+,\mathrm{D}}$ and $\phi_1 \geq 0$ on $\Gamma^1_{\mathrm{S}+,\mathrm{N}}$, then ϕ_1 is the solution to Problem (SP). Otherwise, go to Step 3.

Step 3. If there exists $\tilde{\Gamma}^2_{\mathrm{S}-,\mathrm{N}} \subset \Gamma^1_{\mathrm{S}-,\mathrm{D}}$ such that $\partial_{\mathrm{n}} \phi_0 > 0$, we define $\tilde{\Gamma}^2_{\mathrm{S}-,\mathrm{D}} := \Gamma^1_{\mathrm{S}-,\mathrm{D}} \backslash \overline{\tilde{\Gamma}^2_{\mathrm{S}-,\mathrm{N}}}$, and if there exists $\tilde{\tilde{\Gamma}}^2_{\mathrm{S}-,\mathrm{D}} \subset \Gamma^1_{\mathrm{S}-,\mathrm{N}}$ such that $\phi_0 > 0$, we define $\tilde{\tilde{\Gamma}}^2_{\mathrm{S}-,\mathrm{N}} := \Gamma^1_{\mathrm{S}-,\mathrm{N}} \backslash \overline{\tilde{\tilde{\Gamma}}^2_{\mathrm{S}-,\mathrm{D}}}$, and then define $\Gamma^2_{\mathrm{S}-,\mathrm{D}} := \tilde{\Gamma}^2_{\mathrm{S}-,\mathrm{D}} \cup \tilde{\tilde{\Gamma}}^2_{\mathrm{S}-,\mathrm{D}}$ and $\Gamma^2_{\mathrm{S}-,\mathrm{N}} := \tilde{\Gamma}^2_{\mathrm{S}-,\mathrm{N}} \cup \tilde{\tilde{\Gamma}}^2_{\mathrm{S}-,\mathrm{N}}$. Proceed in the same way to define $\Gamma^2_{\mathrm{S+,D}}$ and $\Gamma^2_{\mathrm{S+,N}}$. Then solve

$$\left\{\begin{array}{rcll} -\Delta\phi_2+\phi_2 &=& f & \text{in } \Omega,\\ \partial_{\mathrm{n}}\phi_2 &=& 0 & \text{on } \Gamma_{\mathrm{N}},\\ \phi_2 &=& 0 & \text{on } \Gamma_{\mathrm{D}},\\ \phi_2 &=& 0 & \text{on } \Gamma^2_{\mathrm{S-,D}}\subset\Gamma_{\mathrm{S-}},\\ \partial_{\mathrm{n}}\phi_2 &=& 0 & \text{on } \Gamma^2_{\mathrm{S-,N}}\subset\Gamma_{\mathrm{S-}},\\ \phi_2 &=& 0 & \text{on } \Gamma^2_{\mathrm{S+,D}}\subset\Gamma_{\mathrm{S+}},\\ \partial_{\mathrm{n}}\phi_2 &=& 0 & \text{on } \Gamma^2_{\mathrm{S+,N}}\subset\Gamma_{\mathrm{S+}}. \end{array}\right.$$

If $\partial_{\mathrm{n}}\phi_2\leq 0$ on $\Gamma^2_{\mathrm{S-,D}}$, $\phi_2\leq 0$ on $\Gamma^2_{\mathrm{S-,N}}$, $\partial_{\mathrm{n}}\phi_2\geq 0$ on $\Gamma^2_{\mathrm{S+,D}}$ and $\phi_2\geq 0$ on $\Gamma^2_{\mathrm{S+,N}}$, then ϕ_2 is the solution to Problem (SP). Otherwise repeat this step.

Iterative switching algorithm revisited for the Tresca-type problem. In this section we adapt the above strategy to the Tresca-type problem (TP) given by

$$\left\{\begin{array}{rl} -\Delta u+u=f & \text{in } \Omega,\\ |\partial_{\mathrm{n}}u|\leq 1 \text{ and } u\partial_{\mathrm{n}}u=-|u| & \text{on } \Gamma. \end{array}\right. \tag{TP}$$

Roughly speaking the algorithm is based on the fact that we need $u=0$ or $|\partial_{\mathrm{n}}u|=1$ on Γ. The main idea is to impose Neumann or Dirichlet boundary conditions on Γ and then to check if the Tresca conditions are satisfied. Otherwise we change the Dirichlet boundary into a Neumann boundary and conversely. More precisely we use the following strategy.

Step 1. We solve the following well-posed problem

$$\left\{\begin{array}{rcll} -\Delta\phi_0+\phi_0 &=& f & \text{in } \Omega,\\ \phi_0 &=& 0 & \text{on } \Gamma. \end{array}\right.$$

If $|\partial_{\mathrm{n}}\phi_0|\leq 1$ on Γ, then ϕ_0 is the solution to Problem (TP). Otherwise we pursue Step 2.

Step 2. If there exists $\Gamma^1_{\mathrm{S-,N}}\subset\Gamma$ such that $\partial_{\mathrm{n}}\phi_0<-1$ and/or if there exists $\Gamma^1_{\mathrm{S+,N}}\subset\Gamma$ such that $\partial_{\mathrm{n}}\phi_0>1$, we define $\Gamma^1_{\mathrm{D}}:=\Gamma\backslash\overline{\Gamma^1_{\mathrm{S-,N}}\cup\Gamma^1_{\mathrm{S+,N}}}$. Then we solve

$$\left\{\begin{array}{rcll} -\Delta\phi_1+\phi_1 &=& f & \text{in } \Omega,\\ \phi_1 &=& 0 & \text{on } \Gamma^1_{\mathrm{D}},\\ \partial_{\mathrm{n}}\phi_1 &=& -1 & \text{on } \Gamma^1_{\mathrm{S-,N}},\\ \partial_{\mathrm{n}}\phi_1 &=& 1 & \text{on } \Gamma^1_{\mathrm{S+,N}}. \end{array}\right.$$

If $|\partial_{\mathrm{n}}\phi_1|\leq 1$ on Γ^1_{D} and $\phi_1\partial_{\mathrm{n}}\phi_1=-|\phi_1|$ on $\Gamma^1_{\mathrm{S-,N}}\cup\Gamma^1_{\mathrm{S+,N}}$, then ϕ_1 is the solution to Problem (TP). Otherwise we pursue Step 3.

Step 3. If there exists $\tilde{\Gamma}^2_{\mathrm{S-,N}}\subset\Gamma^1_{\mathrm{D}}$ such that $\partial_{\mathrm{n}}\phi_1<-1$ and/or if there exists $\tilde{\Gamma}^2_{\mathrm{S+,N}}\subset\Gamma^1_{\mathrm{D}}$ such that $\partial_{\mathrm{n}}\phi_1>1$, we define $\tilde{\Gamma}^2_{\mathrm{D}}:=\Gamma^1_{\mathrm{D}}\backslash\overline{\tilde{\Gamma}^2_{\mathrm{S-,N}}\cup\tilde{\Gamma}^2_{\mathrm{S+,N}}}$. If there exists $\tilde{\tilde{\Gamma}}^2_{\mathrm{D}}\subset\Gamma^1_{\mathrm{S-,N}}$ such that $\phi_1\partial_{\mathrm{n}}\phi_1\neq-|\phi_1|$ (that is, $\phi_1<0$), we define $\tilde{\tilde{\Gamma}}^2_{\mathrm{S-,N}}:=\Gamma^1_{\mathrm{S-,N}}\backslash\overline{\tilde{\tilde{\Gamma}}^2_{\mathrm{D}}}$, and if there exists $\tilde{\tilde{\tilde{\Gamma}}}^2_{\mathrm{D}}\subset\Gamma^1_{\mathrm{S+,N}}$ such that $\phi_1\partial_{\mathrm{n}}\phi_1\neq-|\phi_1|$ (that is, $\phi_1>0$), we define $\tilde{\tilde{\Gamma}}^2_{\mathrm{S+,N}}:=$

$\Gamma^1_{\mathrm{S}+,\mathrm{N}} \backslash \overline{\tilde{\tilde{\tilde{\Gamma}}}^2_{\mathrm{D}}}$. Then we define $\Gamma^2_{\mathrm{D}} := \tilde{\Gamma}^2_{\mathrm{D}} \cup \tilde{\tilde{\Gamma}}^2_{\mathrm{D}} \cup \tilde{\tilde{\tilde{\Gamma}}}^2_{\mathrm{D}}$, $\Gamma^2_{\mathrm{S}-,\mathrm{N}} := \tilde{\Gamma}^2_{\mathrm{S}-,\mathrm{N}} \cup \tilde{\tilde{\Gamma}}^2_{\mathrm{S}-,\mathrm{N}}$ and $\Gamma^2_{\mathrm{S}+,\mathrm{N}} := \tilde{\Gamma}^2_{\mathrm{S}+,\mathrm{N}} \cup \tilde{\tilde{\Gamma}}^2_{\mathrm{S}+,\mathrm{N}}$. Then we solve

$$\left\{ \begin{array}{rcll} -\Delta\phi_2 + \phi_2 & = & f & \text{in } \Omega, \\ \phi_2 & = & 0 & \text{on } \Gamma^2_{\mathrm{D}}, \\ \partial_{\mathrm{n}}\phi_2 & = & -1 & \text{on } \Gamma^2_{\mathrm{S}-,\mathrm{N}}, \\ \partial_{\mathrm{n}}\phi_2 & = & 1 & \text{on } \Gamma^2_{\mathrm{S}+,\mathrm{N}}. \end{array} \right.$$

If $|\partial_{\mathrm{n}}\phi_2| \leq 1$ on Γ^2_{D} and $\phi_2\partial_{\mathrm{n}}\phi_2 = -|\phi_2|$ on $\Gamma^2_{\mathrm{S}-,\mathrm{N}} \cup \Gamma^2_{\mathrm{S}+,\mathrm{N}}$, then ϕ_2 is the solution to Problem (TP). Otherwise we repeat this step.

We emphasize that the proof of convergence of the switching algorithm given in [2] for the Signorini-type problem cannot be easily adapted to the Tresca-type problem. Indeed this proof is based on the strict decrease of the size of the Dirichlet boundary (where $u = 0$) and the strict growth of the Neumann boundary (where $\partial_{\mathrm{n}}u = 0$). For the Tresca-type problem, the main obstacle comes from the fact that there are two different Neumann boundaries (that is, $\partial_{\mathrm{n}}u = 1$ and $\partial_{\mathrm{n}}u = -1$) and thus, to the best of our knowledge, one cannot conclude in a similar way. Hence the proof of convergence of this algorithm for the Tresca-type problem, which is not the heart of the present work, is postponed for a future work.

We conclude this paragraph by implementing the above algorithm on the example of Tresca-type problem considered in Section 4 (for which the explicit expression of the exact solution u_0 is known). Figure 7 represents the exact solution on one hand, and the initial and approximated solutions from the switching algorithm on the other hand. We obtain a H^1-error equal to 0.00403978.

Final remarks on the implementation and on the stopping criterion. In order to numerically solve the problems involved in the above switching algorithms, any finite elements method can be used (note that these problems are standard, without any inequality or nonsmooth boundary condition). In Section 4 we performed simulations using Freefem++ software (see [18]). Let us mention that the above switching algorithms stop when all the boundary conditions are satisfied. We can also add a criterion on the number of iterations. Indeed, in practice, we noticed that these algorithms need only few iterations to converge. In our simulations we impose a maximal number of iterations equal to 10.

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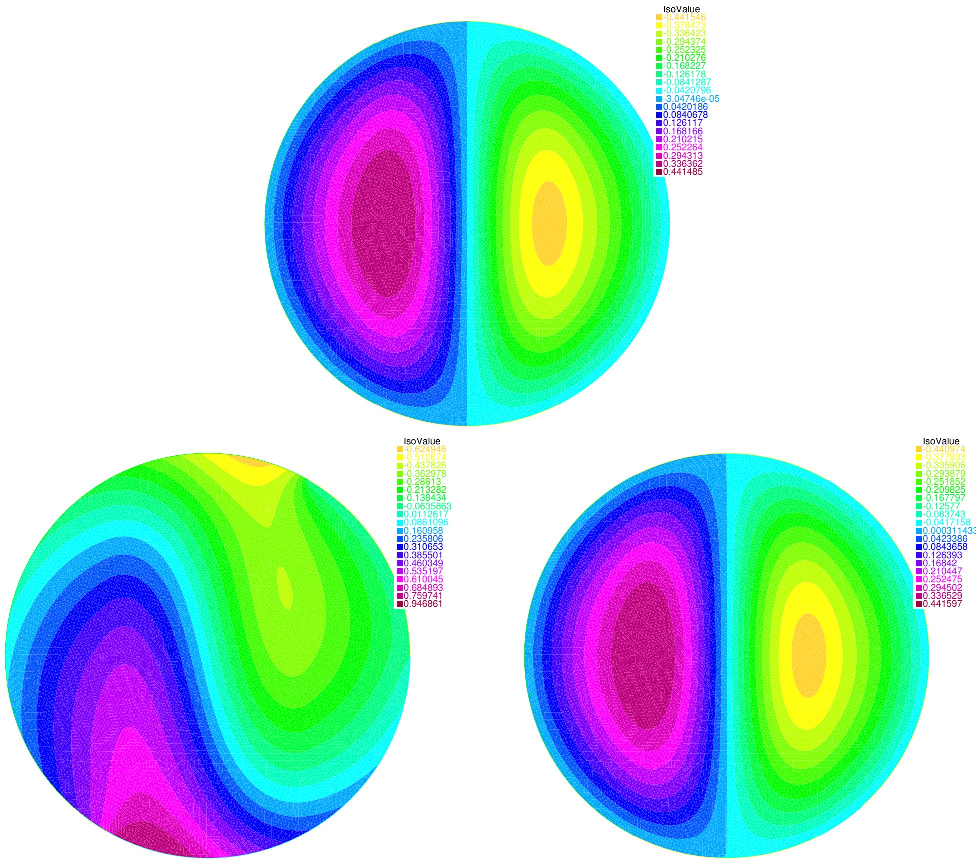

Figure 7: Resolution of a Tresca-type problem: exact solution (top), initial solution (bottom left) and approximated solution (bottom right).

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